SIAG on Optimization Views and News

A Forum for the SIAM Activity Group on Optimization

Volume 32 Number 1 December 2024

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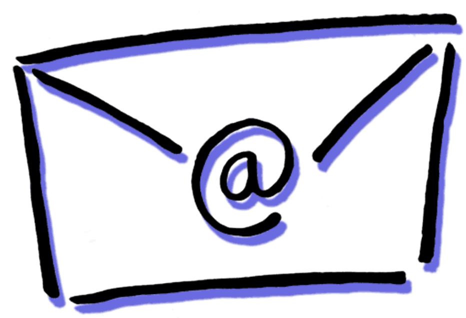

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Articles

Maximal quadratic-free sets: constructions, characterizations, and challenges

Gonzalo Muñoz
Industrial Engineering Department
Universidad de Chile
Beauchef 851, Santiago, Chile
`gonzalo.m@uchile.cl`
`gonzalomunoz.org`

Generating valid inequalities for a non-convex set S is a highly challenging and important task in optimization. Recently, many articles have tackled this endeavor by exploiting the flexibility of the intersection cut framework. Through intersection cuts, one can transform the generation of a cutting plane into the construction of an S-free set, that is, a convex set whose interior does not intersect S. In this article, we summarize recent efforts in S-free set construction tailored to the case where S is defined using a non-convex quadratic inequality. We show the basic constructions behind these sets, computational experiments, recent characterizations and extensions, and current challenges. This article describes joint work with Antonia Chmiela, Joseph Paat, and Felipe Serrano.

1 Introduction

In the last few decades, we have seen a consistent increase in the development of new methodologies for non-convex optimization that can solve challenging and important models in science and engineering. One sign of this has been the growth in software that can solve non-convex optimization models to global optimality (e.g., [17, 6, 10, 22, 23, 28, 32]).

Most of the current optimization methodologies that can solve non-convex problems to global optimality are based, at a high level, on the framework that has proved successful in integer linear programming: in particular outer-approximation techniques that can provide strong dual bounds, heuristics that can find good feasible solutions, and branching. While we have made significant progress, we still encounter multiple challenges that prevent us from solving many non-convex optimization instances to provable optimality. For an overview of the field, we refer the reader to [1, 24, 35, 37].

In this article, we review one stream of work that has focused on producing convexification techniques for non-convex quadratically constrained optimization. This convexification is achieved via *cutting planes*, i.e., valid inequalities that can progressively improve a starting outer-approximation of the feasible region. How to generate cutting planes is usually a challenging question; here, we make use of the flexibility of the *intersection cut* framework. This overview covers the articles [14, 13, 29, 30].

Before diving into the details of this line of work, we note that there have been several articles using the intersection cut framework in non-linear, non-convex settings. For example, this framework has been applied to bilevel optimization [18], polynomial optimization [8, 9], factorable mixed-integer non-linear programs (MINLPs) [34], problems with bilinear terms [19], signomial programming [38], and submodular maximization [39]. Alternative approaches to generate valid inequalities in the quadratic setting can be seen in [12, 25, 33]. We refer to the survey [11] and the references therein for other efforts of extending cutting planes to the non-linear setting.

1.1 Intersection cuts in a nutshell

Consider a generic optimization problem

$$\min\{\bar{c}^\top s \,:\, s \in S \subseteq \mathbb{R}^p\} \tag{1}$$

where S is a closed set, not necessarily convex. Given a linear outer-approximation of S, the *intersection cut* paradigm proceeds as follows. We first solve a *linear programming* (LP) relaxation of (1) and obtain a vertex solution $\bar{s}$. If we are lucky and $\bar{s} \in S$, the problem is solved. If not, we construct a simplicial conic relaxation $K \supseteq S$ with apex $\bar{s}$ and an S-free set C such that $\bar{s} \in \operatorname{int}(C)$. We define an S-free set as a full-dimensional closed convex set satisfying $\operatorname{int}(C) \cap S = \emptyset$. With these ingredients, we can find a cutting plane that separates $\bar{s}$ from S. In Figure 1(left) we show a simple intersection cut in the case where all p rays of K intersect the boundary of the S-free set C. In such a case, the intersection cut is simply defined by the hyperplane containing all such intersection points (hence its name). For more details on the intersection cut framework, see [3, 21, 36].

Constructing a simplicial conic relaxation K is easy, provided we already have a linear relaxation of (1): one can simply take p linearly independent constraints that are active for $\bar{s}$ from the simplex tableau. Constructing the S-free set C, however, can be challenging. This is one of the crucial aspects of the intersection cut framework; it channels the cutting plane generation in a very generic setting into the construction of an S-free set and the computation of the intersection points.

It can be proved that if two S-free sets C, C' are such that $C \subseteq C'$, the intersection cut derived from C' is at least as strong than the one derived from C [16, Remark 6.6][1]. This behavior is shown in Figure 1(right). This makes

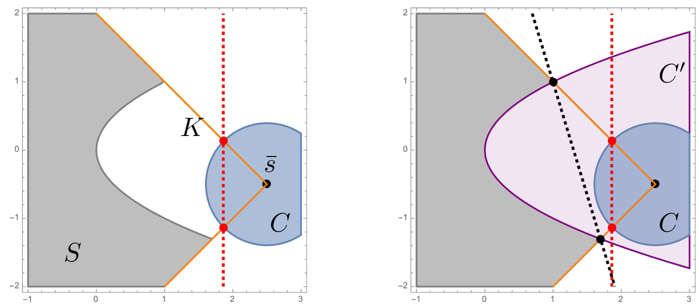


Figure 1: Left: an intersection cut (red) separating $\bar{s}$ from S (gray). The cut is computed using the intersection points of an S-free set C (blue) and the rays of a simplicial cone $K \supseteq S$ (boundary in orange) with apex $\bar{s} \notin S$. Right: the effect of using another S-free set $C' \supsetneq C$. The resulting intersection cut is shown in black. Figure obtained from [14].

inclusion-wise *maximality* of an S-free set a desirable goal to obtain strong cuts.

Once an S-free set C is constructed, the cut can be computed as follows. It is well known that we can assume C is described as $C = \{s \in \mathbb{R}^p \,:\, \phi(s - \bar{s}) \le 1\}$ where ϕ is a sublinear function (e.g. ϕ can be chosen as the gauge of $C - \bar{s}$ [31]). Further assuming without loss of generality that the LP relaxation of (1) is in standard form, we can consider the constraint $\bar{s} + \sum_{i=1}^p r^i s_i \in S$ where $r^i \in \mathbb{R}^p$ are the extreme rays of K and $s_i \in \mathbb{R}_+$. Under these considerations, the intersection cut separating $\bar{s}$ can be described as

$$\sum_{i=1}^p \phi(r^i) s_i \ge 1. \tag{2}$$

A common interpretation of $\phi(r^i)$ is via step lengths: when ϕ is the gauge of $C - \bar{s}$, we have

$$\phi(r) = \inf\left\{\tau \,:\, \bar{s} + \frac{r}{\tau} \in C,\, \tau > 0\right\}.$$

In this case, $1/\phi(r)$ is the step length required to leave C from $\bar{s}$ in the direction r. This means that each vector $\bar{s} + r^i/\phi(r^i)$ is on the boundary of C and defines one of the intersection points we mentioned earlier (and depicted in Figure 1). In general, it is useful to keep this interpretation in mind; however, in a later section we will leverage other ϕ functions that do not correspond to the gauge.

1.2 Maximal quadratic-free sets

The stream of work we overview here deals with problems that are quadratically constrained. That is, we assume S in (1) has the form

$$S = \{s \in \mathbb{R}^p \,:\, s^\top Q_i s + b_i^\top s + c_i \le 0,\, i = 1, \ldots, m\}.$$

where Q_i are symmetric matrices that may or may not be positive semi-definite. Note that if we have $\bar{s} \notin S$, there exists $i \in \{1, \ldots, m\}$ such that

$$\bar{s} \notin S_i := \{s \in \mathbb{R}^p \,:\, s^\top Q_i s + b_i^\top s + c_i \le 0\},$$

[1] This citation deals with S being the lattice, but the argument extends trivially to any closed S.

and constructing an S_i-free set containing $\bar{s}$ suffices to ensure separation. We refer to these S_i-free sets as *quadratic-free sets*. In what follows, we show how to construct *maximal* such sets, computational evaluations and improvements, and recent characterizations of these sets.

1.3 A canonical representation of a quadratic inequality

Slightly abusing notation, we let S refer to a set defined using a single generic quadratic inequality:

$$S = \{s \in \mathbb{R}^p \,:\, s^\top Q s + b^\top s + c \le 0\}. \tag{3}$$

We begin by simplifying the description of the quadratic inequality. Firstly, we note that when Q is positive semidefinite or negative semidefinite, the construction of maximal S-free sets is easy: in the former, they can be obtained from supporting hyperplanes, and in the latter, the complement of S is the unique maximal S-free set. Therefore, we focus on the case where Q is indefinite.

As noted in [30], we can use a linear transformation on S to shift our attention to one of the following two sets:

$$S^h := \{(x, y) \in \mathbb{R}^{n+m} \,:\, \|x\| \le \|y\|\}, \tag{4}$$

$$S^g := \{(x, y) \in \mathbb{R}^{n+m} \,:\, \|x\| \le \|y\|,\, a^\top x + d^\top y = -1\}, \tag{5}$$

where $\|\cdot\|$ is the Euclidean norm and $\max\{\|a\|, \|d\|\} = 1$. The linear transformation is a composition of a homogenization (if needed), a diagonalization, and a projection. Whether S gets mapped to S^h or S^g depends on whether the quadratic defining S is homogeneous, i.e., if $b = 0$ and $c = 0$ in (3), then S is mapped to S^h, otherwise, S is mapped to S^g.

Using this representation, one can derive an easy S^h-free set: by the Cauchy-Schwarz inequality, for any $\lambda \in \mathbb{R}^n$ such that $\|\lambda\| = 1$, we have

$$S^h \subseteq \{(x, y) \,:\, \lambda^\top x \le \|y\|\}.$$

Therefore,

$$C_\lambda := \{(x, y) \in \mathbb{R}^{n+m} \,:\, \|y\| \le \lambda^\top x\} \tag{6}$$

is an S^h-free set. We will see below that this set is maximal, but first, we need to establish a maximality criterion on which to rely.

2 Maximality criteria

In this section, we show the main maximality criteria we have developed. We note that these criteria do not assume that S is a quadratically-defined set; they apply in general.

To motivate our first maximality criterion, let us consider the case when S is a lattice. In this case, Lovász [26] shows that full-dimensional maximal S-free sets are polyhedra with integer points in the relative interior of each facet; see also [2]. We show such a set in Figure 2. In this case, each lattice point in the relative interior of a facet acts as a "certificate" for that facet. Note that being in the interior of a facet means that no other inequality is active at that point. To move to a general S, we rely on this last interpretation.

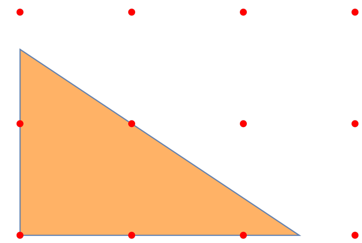


Figure 2: Example of a maximal lattice-free set. The set is lattice-free since there is no integer point in its interior. Furthermore, it is maximal since each facet has at least one integer point in its relative interior.

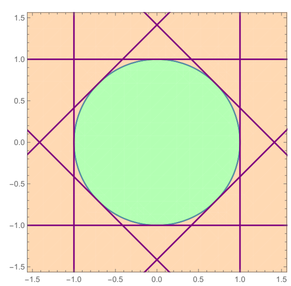


Figure 3: Example of maximality criterion using exposing points. The set S (orange) is reverse-convex, and each purple line defines one valid inequality for the S-free set C (green). There are infinitely many of them. Each point $x \in S \cap C$ is an exposing point for the inequality defined by the purple line passing through x.

Definition 1 (Exposing point, [30])**.** *Given a convex set $C \subseteq \mathbb{R}^n$ and a valid inequality $\alpha^\top x \le \beta$, we say that a point $x_0 \in \mathbb{R}^n$* exposes *$\alpha^\top x \le \beta$ provided both 1) $\alpha^\top x_0 = \beta$ and 2) if $\gamma^\top x \le \delta$ is any other non-trivial valid inequality for C such that $\gamma^\top x_0 = \delta$, then $\gamma^\top x \le \delta$ is a scaled version of $\alpha^\top x \le \beta$. We also say that $\alpha^\top x \le \beta$ is* exposed *by x_0 or that x_0 is an* exposing point *of the inequality.*

The term "exposing" comes from the concept of *exposed points* from convex analysis. If C is convex with 0 in its interior and the valid inequality $\alpha^\top x \le 1$ has an *exposing point*, then α is an *exposed point* of the polar of C.

When every inequality in a description of C has an exposing point, the same phenomenon as in the lattice case is captured: the exposing points act as a certificate for, loosely speaking, not being able to "grow the set in that direction".

Theorem 1 ([30])**.** *Let $S \subseteq \mathbb{R}^n$ be a closed set and let $C \subseteq \mathbb{R}^n$ be an S-free set. Assume that $C = \{x \in \mathbb{R}^n \,:\, \alpha^\top x \le \beta, \forall (\alpha, \beta) \in I\}$ for some $I \subseteq \mathbb{R}^n \times \mathbb{R}$ such that for every $(\alpha, \beta) \in I$ there is an $x \in S \cap C$ that exposes $\alpha^\top x \le \beta$. Then, C is a maximal S-free set.*

For an illustration of this criterion, see Figure 3.

We will see below that this *sufficient* criterion works as a good starting point. However, many maximal S-free sets do not admit exposing points even in the quadratic setting; this phenomenon does not appear in the lattice case. For

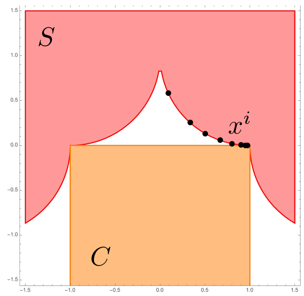


Figure 4: Example of a set S (red), a maximal S-free set C (orange), and an exposing sequence $(x^i)_{i=1}^{\infty}$ (black) for the "horizontal" inequality of C. The exposing sequence is contained in S, and every sequence of hyperplanes separating x^i from C converges to the desired inequality. Figure obtained from [29].

this reason, in [29], a necessary and sufficient criterion was developed.

Before showing the criterion, let us motivate it with an example. Consider Figure 4, where a set S and a maximal S-free set are displayed. In this example, the set C intersects S in the top corners of the rectangle, where two facets of C are active. Thus, there are no exposing points. However, note that the sequence of black points $(x^i)_{i=1}^{\infty}$ approaches the "horizontal facet" of C in a special way: *any* sequence of hyperplanes that separate each x^i converges to the horizontal facet. Therefore, this sequence can act as a more flexible certificate than exposing points. This motivates the following definition.

Definition 2 (Exposing sequence, [29])**.** *Let $C \subseteq \mathbb{R}^n$ be a convex set and let $\alpha^\top x \leq \alpha_0$, with $\alpha \neq 0$, be a valid inequality for C. A sequence $(x^i)_{i=1}^{\infty}$ in $\mathbb{R}^n$ is an* **exposing sequence** *for $\alpha^\top x \leq \alpha_0$ if $\lim_{i\to\infty}(\delta^i, \delta_0^i) = (\alpha, \alpha_0)$ for every sequence $((\delta^i, \delta_0^i))_{i=1}^{\infty}$ in $\mathbb{R}^n \times \mathbb{R}$ such that $\|\delta^i\| = \|\alpha\|$, ${\delta^i}^\top x \leq \delta_0^i$ is a valid inequality for C, and ${\delta^i}^\top x^i \geq \delta_0^i$ for each i.*

We note that when an inequality has an exposing point x_0, the constant sequence $x^i = x_0$ for all i is an exposing sequence.

Theorem 2 ([29])**.** *Let $S \subseteq \mathbb{R}^n$ be a closed set and let $C \subseteq \mathbb{R}^n$ be an S-free set. C is a maximal S-free set if and only if there exists a set $I \subseteq \mathbb{R}^n \times \mathbb{R}$ such that*

$$C = \{x \in \mathbb{R}^n : \alpha^\top x \leq \alpha_0 \ \ \forall\, (\alpha, \alpha_0) \in I\}$$

and each $(\alpha, \alpha_0) \in I$ has an exposing sequence $(x^i)_{i=1}^{\infty}$ in S.

With the maximality criteria laid out, we can begin to certify maximality of our quadratic-free sets.

3 The first maximal quadratic-free sets

In (6), we constructed the S^h-free set

$$C_\lambda = \{(x, y) \in \mathbb{R}^{n+m} \,:\, \|y\| \leq \lambda^\top x\}.$$

To prove the maximality of C_λ, we rely on the criterion of Theorem 1. Note that an outer description of the set C_λ is the following:

$$C_\lambda = \{(x, y) \in \mathbb{R}^{n+m} \,:\, \beta^\top y \leq \lambda^\top x,\ \forall \beta,\ \|\beta\| = 1\}. \qquad (7)$$

We show that, for each inequality in (7), the point (λ, β) is an exposing point. Indeed, $(\lambda, \beta) \in C_\lambda \cap S^h$ since $\|\lambda\| = \|\beta\| = 1$. The inequality $\beta^\top y \leq \lambda^\top x$ is clearly active at $(x, y) = (\lambda, \beta)$. Moreover, for $\beta' \neq \beta$ we have

$$(\beta')^\top \beta < 1 = \lambda^\top \lambda.$$

Hence, no other inequality is tight for (λ, β), and thus the latter is an exposing point. We have proved the following.

Theorem 3 ([30])**.** *The set C_λ is a maximal S^h-free set.*

In Figure 5 we display the set S^h and C_λ when $n+m \leq 3$.

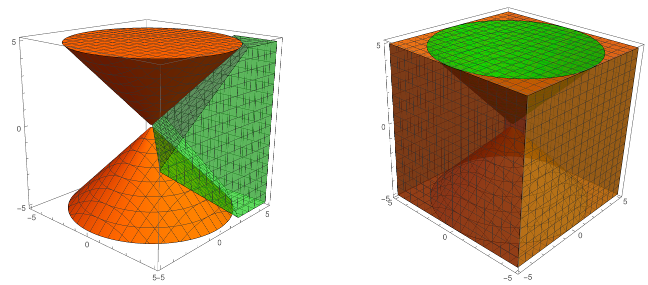


Figure 5: Set S^h (orange) and maximal S^h-free set C_λ (green). On the left, we plot an example with $n = 1$, $m = 2$. On the right, we plot an example with $n = 2$, $m = 1$.

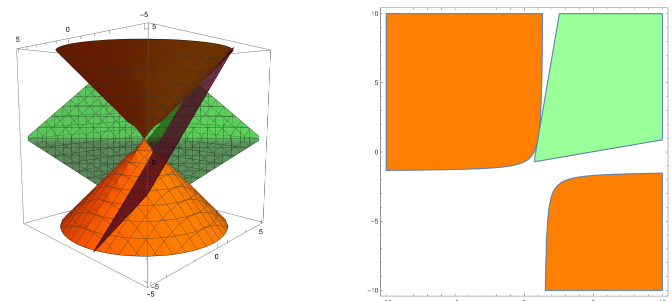


Figure 6: Left: the set S^h (orange), a maximal S^h-free set (green), and the hyperplane H. Right: the same sets displayed on the slice given by H, i.e., $S^h \cap H$ (orange) and $C_\lambda \cap H$ (green).

To move to the non-homogeneous case, one can note that $S^g = S^h \cap H$, where $H := \{(x, y) \in \mathbb{R}^{n+m} \,:\, a^\top x + d^\top y = -1\}$, and thus, $C_\lambda \cap H$ is always S^g-free. This provides a good starting point, but unfortunately, the maximality of an S-free set is not maintained when taking slices. We illustrate this phenomenon in Figure 6.

As can be seen in Figure 6, some inequalities may already have an exposing point in S^g, which means, intuitively, that we cannot enlarge the set in that direction. Other inequalities, however, can be tilted and displaced. We show a maximal S^g-free set in Figure 7, which is obtained from $C_\lambda \cap H$ by enlarging it in this way. The formal construction is quite

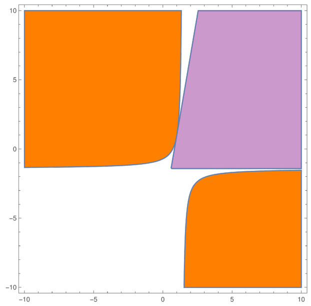


Figure 7: The set $S^g = S^h \cap H$ (orange) and a maximal S^g-free set (purple), obtained from enlarging $C_\lambda \cap H$ in Figure 6.

technical, so we only show the main results and refer the reader to [30] for details.

To display the maximal S^g-free sets, we need the following definition.

Definition 3. *Let $a \in \mathbb{R}^n$, $d \in \mathbb{R}^m$ and let λ satisfy $\|\lambda\| = 1$. We define the function $\phi : \mathbb{R}^m \to \mathbb{R}$ as*

$$\phi(y) = \begin{cases} \|y\|, & \lambda^\top a\|y\| + d^\top y \le 0 \\ \sqrt{(\|y\|^2 - (d^\top y)^2)(1 - (\lambda^\top a)^2)} - d^\top y \lambda^\top a, & \textit{otherwise.} \end{cases} \tag{8}$$

Additionally, for technical reasons, we assume we have at hand $(\bar{x}, \bar{y}) \notin S^g$ that we would like to cut off, which satisfies

$$a^\top \bar{x} + d^\top \bar{y} = -1.$$

This is typically available in a cutting-plane framework, since we can add the last linear equality to the linear relaxation. The maximal S^g-free sets we construct have $(\bar{x}, \bar{y})$ in their relative interiors.

Theorem 4 ([30])**.** *Consider a non-convex set S^g defined as in (5), with $\max\{\|a\|, \|d\|\} = 1$, and $(\bar{x}, \bar{y})$ satisfying both $\|\bar{x}\| > \|\bar{y}\|$ and $a^\top \bar{x} + d^\top \bar{y} = -1$. Consider ϕ defined in (8) and let $\lambda = \frac{\bar{x}}{\|\bar{x}\|}$.*

If $\|a\| \le \|d\| = 1$, the set

$$C_\phi := \{(x, y) \in \mathbb{R}^{n+m} : \phi(y) \le \lambda^\top x\} \tag{9}$$

is maximal S^g-free and contains $(\bar{x}, \bar{y})$ in its interior.

If $\|d\| < \|a\| = 1$, the set

$$C_\phi^g := \left\{ (x, y) : \begin{array}{lr} \|y\| \le \lambda^\top x, & \textit{if } \lambda^\top a\|y\| + d^\top y \le 0 \\ \phi\left(y - \frac{d}{1 - \|d\|^2}\right) \le \lambda^\top \left(x + \frac{a}{1 - \|d\|^2}\right), & \textit{otherwise} \end{array} \right\}.$$

is maximal S^g-free and contains $(\bar{x}, \bar{y})$ in its interior.

These are rather technical formulas, but we would like to provide some intuition for the reader. As noted earlier, the starting point for these sets is C_λ, which can be seen in the formulas of C_ϕ and C_ϕ^g. If we observe Figures 6 and 7, we note that one of the inequalities of $C_\lambda \cap H$ (the inequality that has an exposing point on the slice) remained untouched while the other inequality was tilted and displaced. This is what the condition $\lambda^\top a\|y\| + d^\top y \le 0$ in the definition of ϕ captures. When this condition is met, it indicates that some of the inequalities of C_λ have exposing points even after taking slices. The other inequalities need to be modified, which is what the ϕ function does. One important point is that the new "horizontal" inequality that was obtained after tilting and displacing does not intersect the set S^g, and thus does not have an exposing point. In [30], we use a special criterion for proving maximality in this case. This criterion was further refined to obtain Theorem 2.

This provides our first families of maximal quadratic sets for arbitrary quadratic inequalities. Both in the homogeneous and non-homogeneous case, these sets guarantee the separation of any vertex in an LP relaxation that does not satisfy a quadratic constraint.

4 Computational implementation

With this theoretical construction established, in [14], we focused on the implementation details. In particular, we develop explicit formulas that can be used to compute the intersection cuts, starting from S as in (3) and $\bar{s} \notin S$ a vertex of an LP relaxation. We highlight some of the key features next and refer the reader to [14] for details.

- We start from a description of S as in (3), $\bar{s} \notin S$, and K a simplicial conic relaxation of the feasible region with apex $\bar{s}$.
- We consider an extreme ray r of K and show that to compute the desired step length—and thus a cut coefficient—associated with one of the maximal S-free sets discussed above, it suffices to compute the roots of one-dimensional quadratics of the form
$$A_r t^2 + B_r t + C_r - (D_r t + E_r)^2 = 0.$$
- We provide explicit formulas for A_r, B_r, C_r, D_r, E_r that can be directly implemented.

Additionally, we implemented some of the cuts derived from the outer-product-free sets of [9], which can be derived from implied constraints using additional variables. If one uses an extended formulation in QCQPs, a common procedure is to use an extended formulation that defines a matrix of variables X, where each X_{ij} is meant to represent a term $x_i x_j$. Some of the intersection cuts of [9] are derived from the valid conditions

$$X_{i_1,j_1} X_{i_2,j_2} = X_{i_1,j_2} X_{i_2,j_1}. \tag{10}$$

In [14], we interpret each one of these *implied* quadratic equalities as two quadratic inequality constraints and apply the "maximal quadratic-free machinery" to them.

We tested our cutting planes in SCIP 8.0 [7], using instances of the MINLPLib [27]. We selected all non-convex instances with at least one quadratic constraint; this resulted in 705 out of 1625 instances. All experiments were run with

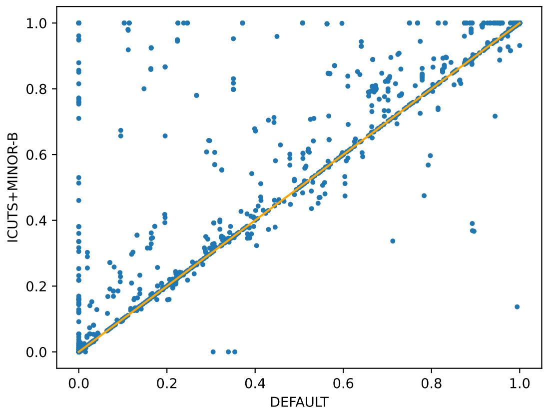


Figure 8: Gap closed comparison in root node experiments between intersection cuts and default SCIP. Points that are above the diagonal are instances where intersection cuts provide better gaps than default SCIP. Figure obtained from [14].

three different permutations for each instance; in our reports, we treat every instance-permutation pair as an individual instance.

We begin presenting the evaluation of our cuts using *root node experiments*: we start from an initial LP relaxation of a QCQP (providing a dual bound d_1) and incorporate the intersection cuts in SCIP using a *separator*. After SCIP stops adding cutting planes, we compute the *gap closed*; that is, if d_2 is the dual bound obtained when the algorithm finishes, and if p is a reference primal bound, then we define the function $GC(p, d_1, d_2) = (d_2 - d_1)/(p - d_1)$ as the *gap closed* improvement. We used the value of MINLPLib's best primal bounds as p and thus discarded instances for which no primal solution was available, or d_1 could not be found.

In Figure 8, we show the effect of adding both intersection cuts from violated quadratic inequalities in the original variables (`ICUTS`) as well as intersection cuts from the *implied* quadratic inequalities in (10) (`MINOR-B`). In these plots, we see that intersection cuts have a significant impact in closing the root node gap and thus producing a tight relaxation before branching.

While these are positive results, in the same article, we show that directly incorporating these cuts in spatial branch and bound does not produce positive results. The main reason is that the cuts have a negative impact on SCIP's primal heuristics. To isolate this effect, we provide experiments where we turn off all heuristics and feed SCIP the optimal solution of an instance. These results are summarized in Table 1.

In Table 1 we see that, on average, the intersection cuts reduce the running time by 1% and the number of processed nodes by 7%; these improvements are 3% and 12%, respectively, for the affected instances. On the subset [1000, 3600] containing the hardest instances, we obtain an improvement of 14% in running time while needing 15% fewer nodes. This confirms that our cuts are good for *proving optimality*, especially in challenging instances. The natural next step would be to modify how SCIP handles them within the heuristics. We discuss this challenge in Section 7.

We note that since version 8.0, these cuts have been available in SCIP. To test them, we refer the interested reader to the `nlhdlr/quadratic/` family of parameters in scipopt.org/doc/html/PARAMETERS.php.

5 Monoidal strengthening

One of the natural follow-up questions to [14] was whether one could exploit the integrality of some of the variables in a mixed-integer quadratically-constrained program (MIQCP) to strengthen the intersections cuts since, so far, our cutting planes have been oblivious to this.

In [13], we incorporated such a strengthening procedure to the intersection cuts derived from maximal quadratic-free sets. We achieved this via *monoidal strengthening* [4]. As in the previous section, we limit ourselves to a high-level description of these topics.

Consider the basic intersection cut (2) with C described as $C = \{s \in \mathbb{R}^p : \phi(s - \bar{s}) \leq 1\}$. Monoidal strengthening exploits the fact that some of the s_i may be integer in (2). The main idea is to, again, consider the condition $\bar{s} + \sum_{i=1}^p r^i s_i \in S$ and modify it in the following way. Assume that *all* s_i are integer to simplify notation. This implies that $\bar{s} + \sum_{i=1}^p (r^i + m_i) s_i \in S + \sum_{i=1}^p m_i s_i$ for any $m_i \in \mathbb{R}^p, i = 1, \ldots, p$. The set of points of the form $\sum_{i=1}^p m_i s_i$ form a *monoid* $M = \{m : m = \sum_{i=1}^p m_i s_i, s_i \in \mathbb{Z}_+\}$, that is, M satisfies $0 \in M$ and $M + M = M$. Thus, we obtain a new relation: $f + \sum_{i=1}^p (r^i + m_i) s_i \in S + M$. If C is not only S-free, but also $(S + M)$-free, then we can use the function ϕ on this new relation to generate a new cut. This is summarized in the following result.

Theorem 5 ([4], Theorem 1)**.** *Let M be a monoid such that C is $(S + M)$-free and let $I = \{i \in [p] : s_i \in \mathbb{Z}\}$ be the index set of the integer variables. Then,*

$$\sum_{i \notin I} \phi(r^i) s_i + \sum_{i \in I} \inf_{m \in M} \phi(r^i + m) s_i \geq 1$$

is valid and dominates the intersection cut.

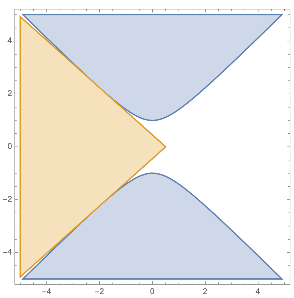
The critical point is to find a monoid M such that C stays $(S + M)$-free. Equivalently, we can also find a monoid M such that (the possibly non-convex) $C - M$ is S-free[2].

In [13], one of the key contributions is the construction of M when S and C satisfy a group of technical assumptions. The intuition for this construction is the following. Consider the maximal S-free set C represented in Figure 9a. The set is maximal because both of its defining inequalities have

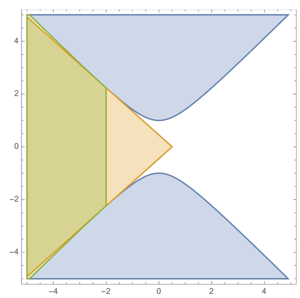
[2]With a slight abuse of notation, we refer to a *non-convex* set $C - M$ as S-free whenever the convex set $C - m$ is S-free for every $m \in M$.

Table 1: Summary of results for branch and bound experiments *without primal heuristics* (optimal solution given to SCIP). Rows labeled $[t, 3600]$ consider instances where one of the settings took at least t seconds. Columns labeled *relative* show the relative improvement of `ICUTS+MINOR-B` with respect to `DEFAULT`. Table obtained from [14].

		`DEFAULT` (no heuristics)			`ICUTS` (no heuristics)			relative	
subset	instances	solved	time	nodes	solved	time	nodes	time	nodes
clean	2034	1224	80.72	2535	1221	79.84	2363	0.99	0.93
affected	660	651	8.36	886	648	8.11	776	0.97	0.88
$[0, 3600]$	1233	1224	5.63	356	1221	5.52	327	0.98	0.92
$[1, 3600]$	591	582	32.05	1710	579	30.73	1499	0.96	0.88
$[10, 3600]$	378	369	109.42	4633	366	104.25	3839	0.95	0.83
$[100, 3600]$	193	184	457.15	22448	181	393.28	19188	0.86	0.86
$[1000, 3600]$	69	60	1528.06	110845	60	1310.26	94217	0.86	0.85



(a) S (blue) with maximal S-free set C (orange). In this case the two inequalities defining C intersect S.



(b) Set of points not in S and "to the left of the exposing points" (green). Note that the green region is not contained in the orange region: see the top left and bottom left.

Figure 9: Construction of the monoid for a maximal S-free set. Figure obtained from [13].

exposing points (Definition 1); the two exposing points of C are the points of the facets of C that are tangent to S.

The monoid M should satisfy that $C - M$ is S-free, and the set $C - M$ is the union of the displacements of C by the elements of $-M$. With this interpretation in mind, we note that a way of translating C in Figure 9a such that the translation is S-free is moving the apex of C to a point not in S and to the left of the exposing points (see Figure 9b). This is the basic idea behind our monoid construction. In [13], we provide the formal definition of the points that are "to the left of the exposing points", and thus define the monoid M explicitly. We also provide proofs that the set is indeed a monoid and that $C - M$ is S-free.

In addition to finding the monoid M, there are two other important questions tackled in [13]. The first question is: how does one find the coefficients of the strengthened inequality defined in Theorem 5? This boils down to solving, for a given ray r, the optimization problem

$$\psi_M(r) := \inf_{m \in M} \phi(r + m). \tag{11}$$

This expression evidently depends on the choice of ϕ, which is a sublinear function that describes the set $C = \{s \in \mathbb{R}^p : \phi(s - \bar{s}) \leq 1\}$. We have mentioned that one can use the gauge function, but there is flexibility in the choice of ϕ function. One strong alternative is using the *minimal representation* of $C - \bar{s}$, that is, the pointwise minimal function ϕ that describes the set as above (see e.g. [5, 15, 40]). However, $\psi_M(r)$ may not be easily computable in this case. In our setting, we provide the minimal representation ϕ of the maximal S-free set we consider and also show how to compute $\psi_M(r)$ (and thus the strengthened coefficients) using this description.

The second question we tackled is related to the uniqueness of lifting procedures. The ψ_M function in (11) provides one way of improving cut coefficients, but there could be other options. In our setting, and leveraging the minimal description ϕ, we show that this lifting is unique, which means that ψ_M gives, in a sense, the best possible coefficients for the integer variables in the intersection cut.

Computationally, we evaluated the advantage of applying monoidal strengthening versus using the basic intersection cuts in branch and bound. We embedded the computation of the monoidal strengthening cut coefficients in SCIP 8.0 as a subroutine of the already implemented intersection cut generator. The test set we considered consists of MINLPLib [27] and QPLib [20]. We selected all non-convex instances with (mixed)-integer constraints and at least one quadratic constraint satisfying the assumptions of the paper; this leaves us with 95 instances. Furthermore, we filtered out all instances that are either infeasible, where no dual bound was found or where monoidal strengthening could not be applied. This left us with a heterogeneous test set of 63 instances, and each instance was run with three different permutations for each instance. As before, we treated every instance-permutation pair as an individual instance.

We considered two different settings that are both based on SCIP's default settings: ICUTS generates the original intersection cuts, whereas MONOIDAL uses the strengthened cutting planes when possible. We restricted ICUTS and MONOIDAL to add at most 20 intersection cuts per quadratic constraint in total.

Summarized results can be found in Table 2. In this table, we observe that MONOIDAL consistently outperforms ICUTS with respect to solving time as well as the number of nodes needed. For example, when looking at the hardest instances (labeled $[1000, 7200]$), MONOIDAL uses 49% less time and 51% fewer nodes than ICUTS. These results show that the proposed monoidal strengthening procedure significantly improves the standard intersection cuts, which highlights the importance of exploiting integrality whenever possible.

However, an important caveat repeats itself in these experiments: our cuts are currently not able to improve the overall performance of default SCIP. As mentioned earlier, our cuts help to obtain better dual bounds but negatively affect SCIP's primal heuristics. Another caveat in this case is that our technical assumptions limit the number of instances where we can apply our strengthening.

At the time of the writing of this article, the latest version of SCIP [10] incorporates the monoidal strengthening procedure for the intersection cuts we described here. To use this procedure, we refer the interested reader to the `nlhdlr/quadratic/` family of parameters in scipopt.org/doc/html/PARAMETERS.php.

6 Every maximal homogeneous-quadratic-free set

The last work that we discuss is related to the task of finding *all* maximal quadratic-free sets. While we have seen that the basic quadratic-free sets we constructed can take us a long way in finding good dual certificates, we believe that finding a complete description of all of them can provide more options when finding cutting planes, which may yield a more successful embedding of the cuts in branch and bound. The first step in this direction was taken in [29], where a description of all maximal quadratic-free sets for homogeneous quadratics was found.

As mentioned in the introduction, a homogeneous quadratic inequality can be transformed into the canonical form $S^h = \{(x, y) \in \mathbb{R}^{n+m} : \|x\| \leq \|y\|\}$. For this set, we previously constructed the maximal S^h-free set $C_\lambda = \{(x, y) \in \mathbb{R}^{n+m} : \|y\| \leq \lambda^\top x\}$. In what follows, we will see that the role of λ can be much more flexible.

Note that we can write S^h as a union of convex sets,

$$S^h = \bigcup_{\beta \in D^m} \underbrace{\{(x, y) \in \mathbb{R}^{n+m} : \|x\| \leq \beta^\top y\}}_{S_\beta},$$

where D^m is the unit sphere in $\mathbb{R}^m$. Therefore, there should be a separating hyperplane between each S_β and a S^h-free set C. Since S_β is a cone, we can assume that this separating hyperplane has the form $\Gamma(\beta)^\top x \geq \beta^\top y$ for some coefficients $\Gamma(\beta)$. Note that Γ induces a function that, for each β, provides some of the coefficients of the separating hyperplane for S_β.

It turns out that we can restrict Γ to take values in the unit sphere. More formally, given $\Gamma : D^m \to D^n$, we define the set

$$C_\Gamma := \{(x, y) \in \mathbb{R}^{n+m} : \Gamma(\beta)^\top x - \beta^\top y \geq 0 \ \ \forall\ \beta \in D^m\}. \tag{12}$$

It is not hard to check that the set C_Γ is S^h-free. Indeed, by the Cauchy-Schwarz inequality, any (x, y) in the interior of C_Γ satisfies $\beta^\top y < \|x\|$ for all $\beta \in D^m$ which implies that $\|y\| < \|x\|$.

Moreover, every full-dimensional maximal S^h-free set can be written in the 'standard form' (12).

Theorem 6 ([29])**.** *Let C be a full-dimensional closed convex maximal S^h-free set. There exists a function $\Gamma : D^m \to D^n$ such that $C = C_\Gamma$.*

In light of Theorem 6, the question of characterizing full-dimensional maximal S^h-free sets can be reduced to characterizing the functions Γ corresponding to maximal C_Γ. Let us deduce a sufficient condition for C_Γ to be maximal in a simple case, that is, the case where every inequality has an exposing point. Fix β'. The point $(x', y') = (\Gamma(\beta'), \beta')$ satisfies

$$\Gamma(\beta')^\top x' - (\beta')^\top y' = 0.$$

Moreover, (x', y') is not tight for the other inequalities if

$$\Gamma(\beta)^\top \Gamma(\beta') - \beta^\top \beta' > 0 \qquad \forall \beta \neq \beta'.$$

Rearranging terms, and using that both β and $\Gamma(\beta)$ have unit norm, this can be rewritten as

$$\|\Gamma(\beta) - \Gamma(\beta')\| < \|\beta - \beta'\|. \tag{13}$$

Therefore, we have proved that if Γ satisfies (13) for every pair $\beta \neq \beta'$, then C_Γ defines a maximal S^h-free set. The set C_λ defined in (6) is a special case of C_Γ: if $\Gamma(\beta) = \lambda$ for all β, then $C_\Gamma = C_\lambda$. However, C_Γ has flexibility in what Γ can be, which gives us significant freedom in the construction of maximal S^h-free sets beyond C_λ.

As another example, consider $n = m = 2$ and define Γ using polar coordinates: for $\theta \in [0, 2\pi]$, we define $\gamma(\theta) = \theta(2\pi - \theta)/(4\pi)$ and define Γ such that

$$\beta(\theta) := (\cos(\theta), \sin(\theta)) \mapsto \Gamma(\beta(\theta)) := (\cos(\gamma(\theta)), \sin(\gamma(\theta))).$$

Table 2: Summary of results for branch-and-bound experiments. Rows labeled $[t, 7200]$ consider instances where one of the settings took at least t seconds. Columns labeled *relative* show the relative improvement of MONOIDAL compared to ICUTS. Table obtained from [13].

		ICUTS			MONOIDAL			relative	
subset	instances	solved	time	nodes	solved	time	nodes	time	nodes
all	189	113	221.87	4914	115	214.63	4782	0.97	0.97
$[0, 7200]$	115	113	22.81	936	115	21.56	883	0.95	0.94
$[1, 7200]$	83	81	67.62	2377	83	62.40	2184	0.92	0.92
$[10, 7200]$	81	79	72.54	2574	81	66.56	2341	0.92	0.91
$[100, 7200]$	23	21	724.66	186545	23	565.24	144747	0.78	0.78
$[1000, 7200]$	10	8	2475.04	631764	10	1252.96	307639	0.51	0.49

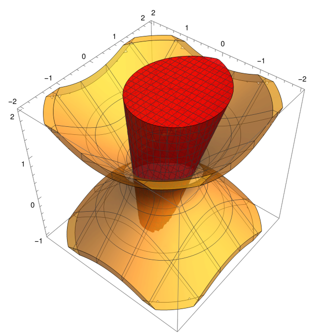


Figure 10: 3-dimensional slices of 4-dimensional sets S^h (boundary in orange) and maximal S^h-free set C_Γ (red). We note that slices may not preserve maximality. Figure obtained from [29].

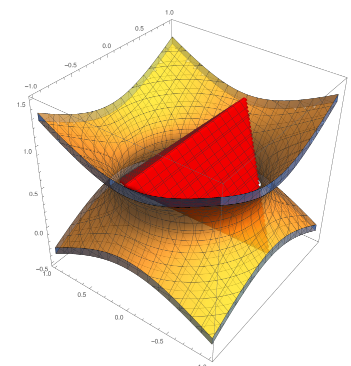


Figure 11: 3-dimensional slice of the 4-dimensional sets S^h (boundary in orange) and a maximal S^h-free set C_Γ (red). Figure obtained from [29].

This definition of Γ satisfies (13) (see [29] for these details), and thus C_Γ is maximal. In Figure 10 we show a 3-dimensional slice of this 4-dimensional C_Γ.

While (13) expands the maximal S^h-free sets we can build, it does not paint the full picture. A full characterization can be obtained by relaxing the strict inequality. We say a function is *non-expansive* if

$$\|\Gamma(\beta) - \Gamma(\beta')\| \leq \|\beta - \beta'\| \qquad \forall \beta, \beta'. \tag{14}$$

Theorem 7 ([29])**.** *Let $\Gamma : D^m \to D^n$ and define C_Γ as in (12). The set C_Γ is a full-dimensional maximal S^h-free set if and only if Γ is non-expansive and $0 \notin \operatorname{conv}(\{\Gamma(\beta) : \beta \in D^m\})$.*

The technical condition $0 \notin \operatorname{conv}(\{\Gamma(\beta) : \beta \in D^m\})$ is mainly needed to ensure full-dimensionality of the set C_Γ, although its role is slightly more subtle and is used not only for full-dimensionality. We also remark that this condition is equivalent to saying that the image of Γ is strictly contained in a hemisphere.

Let us show a nice example that this theorem allows us to construct. Suppose $n = m$ and define $\Gamma(\beta) = |\beta|$, where the absolute value is taken component-wise. The reverse triangle inequality $||a| - |b|| \leq |a - b|$ implies Γ is non-expansive. In addition, for each $\beta \in D^m$, the point $\Gamma(\beta)$ is non-negative and strictly positive in at least one component. Hence, $0 \notin \operatorname{conv}(\{\Gamma(\beta) : \beta \in D^m\})$. Theorem 7 then ensures that C_Γ is a full-dimensional maximal S^h-free set.

It can be shown that this set C_Γ is polyhedral. In fact,

$$C_\Gamma = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : x_i \geq |y_i| \ \ \forall\, i \in \{1, \ldots, m\}\}.$$

Figure 11 illustrates a 3-dimensional slice of the 4-dimensional sets S^h and C_Γ obtained for $n = m = 2$.

One remarkable feature of this set is that it also remains maximal on the 3-dimensional slice shown in Figure 11. This allows us to illustrate the following behavior. It can be seen that $C_\Gamma \cap S^h = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : x_i = |y_i| \ \ \forall\, i \in \{1, \ldots, m\}\}$. Therefore, every facet of C_Γ intersects S^h and, more importantly, any $(x, y) \in C_\Gamma \cap S^h$ is contained in m different facets of C_Γ. Consequently, there is no exposing point in $C_\Gamma \cap S^h$ for any of the facets of C_Γ. This can be observed in the slice as well: no relative interior of a facet intersects S^h. For this reason, maximality in this case cannot be proven using Theorem 1. In this case, we need exposing sequences and Theorem 2. This is the core of the proof of Theorem 7.

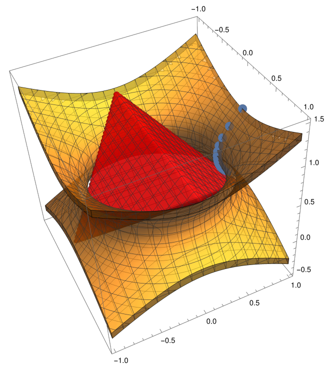


Figure 12: Illustration of how an exposing sequence would certify maximality in the example of Figure 11.

In Figure 12, we illustrate how an exposing sequence would look in this case.

7 Limitations, open questions, and future work

To conclude, we summarize current and future research opportunities of this line of work.

First, we would like to incorporate these cutting planes successfully in a branch and bound method. As mentioned in the computational results, one of the main issues is that our cutting planes negatively affect SCIP's heuristics. We would like to modify how SCIP handles these cuts, so we can get the best of both worlds: better dual bounds without negatively affecting the primal heuristics.

Related to the previous point is the computational incorporation of new families of cutting planes based on the sets C_Γ. The new characterization of maximal S^h-free sets opens the door to a plethora of new cuts, some of which could be either stronger or simply better suited for branch-and-bound. It is unclear at the moment which Γ functions would be preferable from a computational standpoint, and if perhaps the conic relaxation used in the intersection cut computation could guide the construction of the Γ function.

Finally, we are greatly interested in producing more maximal quadratic-free sets for the non-homogeneous setting. Ideally, we would like a characterization akin to that of Theorem 7 for a non-homogenous quadratic inequality. This seems like a challenging task.

Acknowledgments The author would like to thank Antonia Chmiela, Joseph Paat, and Felipe Serrano for their valuable feedback on this article.

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Bulletin

Email items to siagoptnews@lists.mcs.anl.gov *for consideration in the bulletin of forthcoming issues.*

Event Announcements

MOPTA 2025
17-20 June 2025
Ponta Delgada, Açores, Portugal

The *Modeling and Optimization: Theory and Applications* (MOPTA) conference series gathers experts in discrete and continuous optimization. MOPTA, founded by Tamás Terlaky 25 years ago, is the flagship conference of the Industrial and Systems Engineering (ISE) department of Lehigh University. It consists of invited talks and contributed talks across three days.

The 2025 edition of MOPTA will take place in Ponta Delgada, Açores, Portugal on 17-20 June 2025, and will also host the AIMMS/MOPTA Competition. This year's MOPTA will also celebrate the 70th birthday of Professor Terlaky.

URL: https://coral.ise.lehigh.edu/mopta2025

EURO 2025
22-25 June 2025
University of Leeds, UK

EURO 2025 is part of a series of conferences that are the major EURO events. Plenary, semi-plenary, tutorials, and panel sessions will be given by international speakers. This year, the EURO conference will take place at the University of Leeds, UK.

Additionally, the EURO conference will host the awarding conference for EURO prizes: the EURO Distinguished Service Award; the EURO Gold Medal, accompanied by a lecture of the Gold Medal laureate; the EURO Excellence in Practice Award; the EURO Doctoral Dissertation Award; the EURO Prize for OR for the Common Good; and the EURO Award for the Best EJOR Papers.

URL: https://euro2025leeds.uk

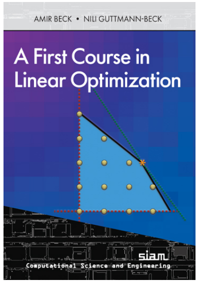
ICCOPT 2025
19-24 July 2025
University of Southern California
Los Angeles, CA

The *International Conference on Continuous Optimization (ICCOPT)* is the flagship conference of the Mathematical Optimization Society in the area of continuous optimization. It is held every three years.

The eighth edition of ICCOPT will take place on the campus of the University of Southern California, in downtown Los Angeles, California. The conference, which is scheduled for July 21-24, 2025, will be preceded by a Summer School on July 19 and 20. The conference will have plenary and semi-plenary talks as well as and parallel sessions with international participants.

URL: https://sites.google.com/view/iccopt2025

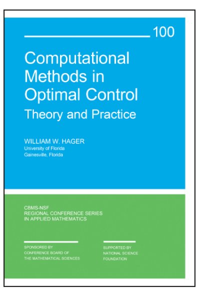
Books



A First Course in Linear Optimization
by Amir Beck and Nili Guttmann-Beck
Publisher: SIAM
ISBN: 978-1-61197-829-2
Published: 2025
Series: Computational Science and Engineering

About the book: This self-contained textbook provides the foundations of linear optimization, covering topics in both continuous and discrete linear optimization. It gradually builds the connection between theory, algorithms, and applications so that readers gain a theoretical and algorithmic foundation, familiarity with a variety of applications, and the ability to apply the theory and algorithms to actual problems.

Audience: This book is for a first undergraduate course in linear optimization, such as linear programming, linear optimization, and operations research. It is appropriate for students in operations research, mathematics, economics, and industrial engineering, as well as those studying computer science and engineering disciplines.



Computational Methods in Optimal Control: Theory and Practice
by William W. Hager
Publisher: SIAM
ISBN: 978-1-61197-825-4
Published: 2025
Series: CBMS-NSF Regional Conference Series in Applied Mathematics

About the book: Using material from many different sources in a systematic and unified way, this self-contained book provides both rigorous mathematical theory and practical numerical insights while developing a framework for determining the convergence rate of discretizations to optimal control problems. Elements of the framework include the reference point, the truncation error, and a stability theory for the linearized first-order optimality conditions.

Within this framework, the discretized control problem has a stationary point whose distance to the reference point is bounded in terms of the truncation error. The theory applies to a broad range of discretizations and provides completely new insights into the convergence theory for discrete approximations in optimal control, including the relationship between orthogonal collocation and Runge-Kutta methods.

Throughout the book, derivatives associated with the discretized control problem are expressed in terms of a backpropagated costate. In particular, the objective derivative of a bang-bang or singular control problem with respect to a switch point of the control are obtained, which leads to the efficient solution of a class of nonsmooth control problems using a gradient-based optimizer.

Audience: Computational Methods in Optimal Control: Theory and Practice is intended for numerical analysts and computational scientists. Users of the software package GPOPS may find the book useful since the theoretical basis for the GPOPS algorithm is developed within the book. It is appropriate for courses in variational analysis, numerical optimization, and the calculus of variations.

Chair's Column

It is with great pleasure that I reconnect with you through this column, since my first SIAG on Optimization Views and News chair's message in 2023. Serving as the chair of such an outstanding community is a tremendous honor, and I am continually inspired by the vibrant energy, dedication, and talent of our members. The SIAM Activity Group on Optimization plays a pivotal role in advancing optimization across diverse domains, fostering groundbreaking research, driving innovation in education, and enabling impactful applications in industry and society. Together, we are shaping the future of optimization, and I am excited to share updates and celebrate our collective achievements in this issue.

And what great news I have to share! I am delighted to announce that the 2026 SIAM Conference on Optimization will take place at the University of Edinburgh, from July 2-5, 2026. Our premier event will feature an exceptional lineup of invited plenary speakers, who will share their insights on cutting-edge developments in optimization. We are also excited to offer two minitutorials designed to introduce foundational and emerging themes to a broad audience. The conference will explore a rich spectrum of topics, ranging from foundational themes to application themes that address pressing challenges. I extend my deepest gratitude to the conference co-chairs, Miguel Anjos and Gabriele Eichfelder, and the members of the organizing committee (program committee in SIAM's terminology) for their invaluable suggestions and advice in shaping this event. A special thanks also goes to the local organizing committee, and its co-chairs Lars Schewe and Miguel Anjos, whose efforts will certainly ensure a seamless experience for all attendees. Finally, I would like to express my heartfelt appreciation to all plenary speakers and minitutorial instructors for their future contributions, which will be central to making this conference a resounding success. Please stay tuned, as announcements regarding minisymposia organization and abstract submission will be made soon.

I wish you all an optimal 2025! Luis

Luis Nunes Vicente (Chair, SIAG on Optimization)
Timothy J. Wilmott Endowed Chair Professor and Department Chair
Department of Industrial and Systems Engineering, Lehigh University

2026 SIAM Conference on Optimization
June 2-5, 2026
University of Edinburgh, UK
https://www.siam.org/conferences-events/siam-conferences/op26

Invited Plenary Talks:

- Radu Ioan Bot, University of Vienna, Austria
- Andrea Lodi, Cornell Tech University, USA
- Ruth Misener, Imperial College London, UK
- Laura Sanità, Bocconi University, Italy
- Ruoyu Sun, Chinese University of Hong Kong, China
- Stefan M. Wild, Lawrence Berkeley National Laboratory, USA

Minitutorials:

- *Performance and computer-added analyses of optimization methods*, François Glineur, UCLouvain, Belgium Adrien B. Taylor, INRIA, France
- *Fair and interpretable resource allocation and machine learning*, Phebe Vayanos, USC, USA

Organizing Committee:

- Miguel Anjos, co-chair, University of Edinburgh, UK
- Gabriele Eichfelder, co-chair, Technische Universität Ilmenau, Germany
- Luis Nunes Vicente, co-chair, Lehigh University, USA
- Heinz Bauschke, University of British Columbia, Canada
- Ana Luisa Custodio, New University of Lisbon, Portugal
- Santanu Subhas Dey, Georgia Institute of Technology, USA
- Susan R. Hunter, Purdue University, USA
- Monique Laurent, CWI and Tilburg University, Netherlands
- Laura Palagi, Sapienza Università di Roma, Italy
- Andrea Raith, University of Auckland, New Zealand
- Chen Xiaojun, The Hong Kong Polytechnic University, Hong Kong
- Wotao Yin, Alibaba Group US/DAMO Academy, USA
- Alain B. Zemkoho, University of Southampton, UK

Local Organizing Committee:

- Lars Schewe, co-chair, University of Edinburgh, UK
- Miguel Anjos, co-chair, University of Edinburgh, UK
- Merve Bodur, University of Edinburgh, UK
- Jacek Gondzio, University of Edinburgh, UK
- Andreas Grothey, University of Edinburgh, UK
- Akshay Gupte, University of Edinburgh, UK

Foundational Themes:

- Conic and linear optimization
- Derivative-free optimization
- Game theory and equilibrium problems
- Geometric perspectives in optimization
- Graphs and network optimization
- Integer optimization
- Nonlinear optimization
- PDE-constrained optimization
- Polynomial and global optimization
- Stochastic and robust optimization
- Variational inequalities and nonsmooth optimization

Application Themes:

- Optimization in data science
- Optimization in health care
- Optimization in energy
- Optimization in control systems
- Optimization in quantum computing
- Optimization in science and engineering
- Optimization in machine learning/AI

Comments from the Editors

We are pleased to present this issue of Views and News. This issue has been published later than usual for several reasons beyond our control, but expect another issue to appear in 2025. In this issue, we feature an article contributed by Gonzalo Muñoz on quadratic-free sets. These sets are of practical interest in solving mixed integer nonlinear optimization problems with nonconvex quadratic constraints.

All issues of *Views and News* are available online at https://siagoptimization.github.io/ViewsAndNews.

May all your algorithms be efficient, and may you always find certificates of global optimality!

The SIAG on Optimization Views and News mailing list, where editors can be reached for feedback, is siagoptnews@lists.mcs.anl.gov. Suggestions for new issues, comments, and papers are always welcome.

Pietro Belotti
DEIB, Politecnico di Milano
Email: pietro.belotti@polimi.it
Web: https://belotti.faculty.polimi.it

Dmitriy Drusvyatskiy
Mathematics Department, University of Washington
Email: ddrusv@uw.edu
Web: https://sites.math.washington.edu/~ddrusv

Matt Menickelly
Argonne National Laboratory
Email: mmenickelly@anl.gov
Web: https://www.mcs.anl.gov/~menickmj