

# SIAG/OPT Views-and-News

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## Contents

### Articles

*Geometric Nonlinear Programming Test Problems*

José Mario Martínez ..... 1

*Extracting Salient Features From Less Data via  $\ell_1$ -Minimization*

Wotao Yin and Yin Zhang ..... 11

**Bulletin** ..... 20

### Chairman's Column

Michael C. Ferris ..... 21

### Comments from the Editor

Luís N. Vicente ..... 21

## Articles

### Geometric Nonlinear Programming Test Problems

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## 1. Introduction

*Convex Nonlinear Programming problems are all alike; every nonconvex problem is difficult in its own way.* I am not the first numerical analyst to borrow the most quoted line of *Anna Karenina* to highlight the difficulties of non-linearity or non-convexity<sup>1</sup>. It could be argued that convex problems are not really “all alike”, but happy families are not either, therefore both the famous first sentence of Tolstoi’s novel and its mathematical corrupted version evoke the way in which diversity of obstacles increases, as far as landscapes become more complex.

Practical optimizers face this dilemma when they need to evaluate nonlinear optimization algorithms. “Normal” theory, based on the behavior of bounded subsequences and on local convergence rates, is not enough to predict efficiency and reliability of a method. As a consequence, many “plausibility” arguments are many times invoked to justify the introduction of new ideas. For example, as in a golf

<sup>1</sup> Professor Teodor Atanacković, from Novi Sad, uses to begin his talks with a variation of Tolstoi’s statement “Happy families are all alike; every unhappy family is unhappy in its own way”.

game, large steps should be taken far from the solution (of course, if we approximately know the correct direction).

Ultimately, Nonlinear Programming algorithms need to be tested. Numerical testing is unavoidable in two different senses. On one hand, testing is needed in the process of development of methods, in order to find and evaluate new algorithmic procedures. Clever examples are the most powerful motivators of useful algorithmic ideas. On the other hand, a reasonably broad set of test problems is needed to corroborate hypotheses on the quality of the algorithm. Collections of problems are also used for comparison purposes.

Roughly speaking, we may distinguish between “Toy Problems” and “Real-life Problems”. The minimization of the banana-like Rosenbrock’s function is the most famous toy problem. This problem was used since the sixties to test the ability of unconstrained minimization solvers to follow efficiently long narrow valleys along their trajectory to the minimizer. Being a problem with only two variables, the level sets of Rosenbrock’s banana are well known, and the problem has been very useful for the improvement of the first numerically efficient unconstrained minimization algorithms. The topological characteristics of many other toy problems (especially many-variable ones) are less known. Of course, the number of possible toy problems is unlimited and their internal classification is very hard. It is easy to define even one-dimensional toy problems that are almost impossible to solve, except using exhaustive enumeration of floating point numbers. The advantage of toy problems is that they are easy to manipulate, simplify or complicate in the process of algorithmic invention.

Real-life problems come from some external application. Physics, Chemistry, Engineering, Biology, Economy and Industry are the most common sources of applied optimization problems. From a pragmatic point of view, the goal of an optimization algorithm is to solve as many real-life problems as possible, or, perhaps, to solve an important subclass of real-life problems in a very efficient way. Unlike toy problems, real-life problems are not easily available (for example, they may depend of a huge amount of data) and are difficult to manipulate.

Strictly speaking, the Geometric Problems<sup>2</sup> reviewed in this paper are toy problems in the sense that they are not necessarily connected to a specific application. However, they have a clear *geometrical meaning* and, because of this meaning, they resemble realistic applications. Their main characteristics may be summarized as follows:

- As many other toy problems and even real-life problems, Geometric Problems are variable-dimensional in several senses. The number of variables and constraints vary according to the dimension of the underlying space, number of “objects” considered and other problem parameters.
- The problems usually have many local minimizers. Finding the global minimum, or a suitable local minimum is a challenging motivation for NLP solvers.
- Geometric Problems are easy to code. Their derivatives are not hard to compute. They do not depend on a huge amount of data.
- Geometric Problems usually admit several Non-linear Programming formulations.
- Failure or success in the solution of geometric problems always has a graphical spatial interpretation. This interpretation provides powerful hints for the improvement of algorithms.

Last, but not least, many people experiment an aesthetic pleasure in solving geometric problems, that surpass the obvious satisfaction of minimizing Rosenbrock’s function up to high precisions. Although this feeling adds nothing to the relevance of the activity, it constitutes a powerful motivation for practical research. As visual animals, we are genetically trained to experiment joyfulness in the presence of spatial elegance and harmony.

## 2. Hard-Spheres

*Hard-Spheres* is the problem of finding  $n_p$  points on the unitary sphere of  $\mathbb{R}^{n_d}$  in such a way that the

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<sup>2</sup>In this article we use the word “Geometric” in its naive sense. We do not mean to address “Geometric Programming” problems with their specific posynomial optimization meaning.

minimal pairwise distance is maximal [9]. Using that  $\|p_i - p_j\|_2^2 = 2 - 2\langle p_i, p_j \rangle$  when  $\|p_i\|_2 = \|p_j\|_2 = 1$ , this problem may be formulated as follows [13]:

$$\begin{aligned} & \text{Minimize } z \\ & \text{subject to} \\ & z \geq \langle p_i, p_j \rangle \quad \text{for all } i \neq j, \\ & \|p_k\|^2 = 1 \quad \text{for all } k = 1, \dots, n_p. \end{aligned} \tag{1}$$

See Figure 1.

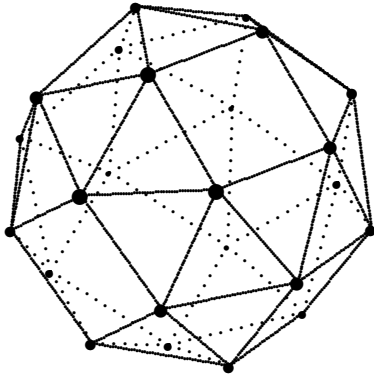


Figure 1: Figure 1: Hard-Spheres  $n_d = 3, n_p = 24$ .

Hard-Spheres has an immediate relation with the famous *Kissing Problem* [9]. Assume that a feasible solution of Hard-spheres (as formulated above) exists such that the objective function value is less than or equal to 0.5. Then, the maximal pairwise distance is not bigger than 1. This means that one can place  $n_p$  non-overlapping solid balls of radius 1/2 touching a given central ball with radius 1/2. The *Kissing Number* associated with  $n_d$  is precisely the maximum number of balls of radius  $R$  that can be placed touching a central ball of the same radius. Therefore, if, at a solution of Hard-Spheres, the objective function value is not greater than 0.5, the Kissing Number of  $n_d$  is not smaller than  $n_p$ . In other words, if we were able to solve all Hard-Spheres problems, all Kissing Problems would be solved too. However, most instances of the Kissing Problem remain open. Clever Semidefinite Programming relaxations have been used for finding Kissing Number bounds [24].

Several different mathematical programming formulations of this problem may be found in [14] and references therein.

The number of inequality constraints of Hard-Spheres grows quadratically with  $n_p$ . Therefore, this number is potentially huge and the number of variables is also huge if one chooses to replace inequality constraints of type  $g_i(x) \leq 0$  by  $g_i(x) + z_i = 0, z_i \geq 0$ , as many NLP solvers do. As a consequence, the problem represents a challenging probation for the ability of solvers to find global minimizers of large-scale problems.

### 3. Maps

Assume that, for all  $j = 1, \dots, n_c$ ,  $\Gamma_j$  is an  $n_j$ -uple of integers between 1 and  $m$  and  $n_j \geq 3$  for all  $j$ . Let  $\beta_j > 0$  for all  $j = 1, \dots, n_c$ . We wish to find points  $p_1, \dots, p_m \in \mathbb{R}^2$  such that the area enclosed by the points  $p_i, i \in \Gamma_j$  is close to  $\beta_j$ , for all  $j = 1, \dots, n_c$ .

Let us be more precise. Suppose that the points  $p_i, i \in \Gamma_j$  are  $(x_1, y_1), \dots, (x_{n_j}, y_{n_j})$ . Then, the “area enclosed” by this points is defined by

$$\text{Area}_j = \frac{1}{2} \left[ (y_1 x_{n_j} - x_1 y_{n_j}) + \sum_{i=1}^{n_j-1} (y_{i+1} x_i - x_{i+1} y_i) \right].$$

If  $(x_1, y_1), \dots, (x_{n_j}, y_{n_j})$  are the consecutive vertices of a simple polygon this formula defines the polygon area. Therefore, this problem is a 2D version of the Multidimensional Scaling problem, where one wants to find points that realise a set of measured distances.

If a set of probable localizations  $\bar{p}_1, \dots, \bar{p}_m$  of the points  $p_1, \dots, p_m$  is given, the problem may be formulated as:

$$\text{Minimize } \sum_{i=1}^m \|p_i - \bar{p}_i\|_2^2$$

subject to

$$0.99\beta_j \leq \text{Area}_j \leq 1.01\beta_j, j = 1, \dots, n_c.$$

This formulation has been used to draw a rough map of America where the areas of the countries are close to the true areas but many other toy problems may be defined and are instructive. In the *America* problem we used  $n_c = 17$  (number of countries),  $m =$

132 (number of points). The “desired localizations”  $\bar{p}_1, \dots, \bar{p}_{132}$  were given by the coordinates of points in a standard map of America. Each set  $\Gamma_j$  defined the indices of the points that belong to the boundary of each country. The difficulty of the problem is, of course, in the fact that the same point may belong to different boundaries.

The solution is given in Figure 2.

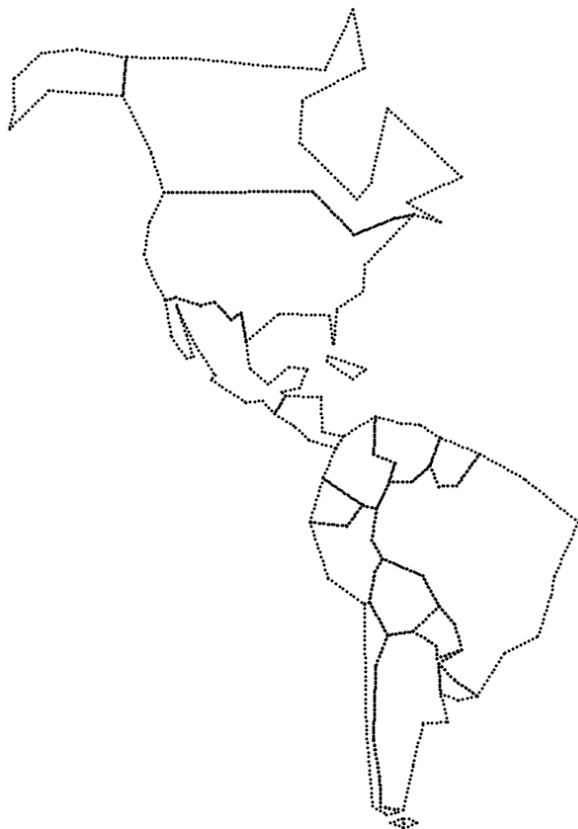


Figure 2: Map of America with true areas.

## 4. Location

We wish to find the point in a given rectangle that minimizes the sum of distances to a set of given polygons (polygons or circles). In addition we impose that neither the selected point nor the points that realise the individual distances in the polygons can belong to a given ellipse. Figure 3 describes the problem and gives the solution for a small case.

The unknowns of this problem are  $p_0$  (the point in the rectangle) and  $p_1, \dots, p_{n_{pol}}$ , the points in the polygons that realise the distances. The polygons

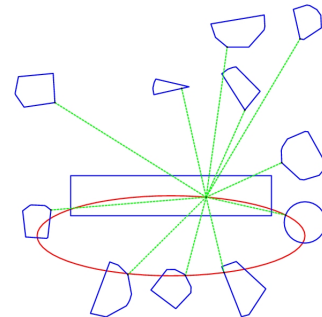


Figure 3: Location problem with 11 polygons.

are described by linear inequality constraints. The ellipse defines  $n_{pol} + 1$  nonconvex inequality constraints.

This problem possesses a structural peculiarity that favors its resolution by means of projection methods, especially in the large-scale case. Projecting a point  $x = (p_0, p_1, \dots, p_{n_{pol}})$  on the feasible region (excluding the constraints defined by the ellipse) is very simple. Taking advantage of this fact, it is possible to solve problems of this type with extremely large dimensions [1, 5].

## 5. Enclosing sets

We wish to find the smallest-volume set of a given class  $\mathcal{C}$  that encloses  $n_p$  given points in  $\mathbb{R}^{n_d}$ . The case in which  $\mathcal{C}$  is the family of ellipsoids has been studied by Todd [23], who gave complete results characterizing global minimizers. Figure 4 shows the solution of a problem in which the points are the atoms of a protein and the ellipsoid is centered in the origin of  $\mathbb{R}^3$ .



Figure 4: Enclosing ellipsoid.

Representing the ellipsoid as the set of points that verify  $(x - a)^T H (x - a) \leq 1$ , we see that the problem above has  $n_d(n_d + 1)/2 + n_d$  unknowns (the coeffi-

cients of the symmetric positive definite Hessian  $H$  and the entries of the center  $a$ ). The constraints are of the form

$$(p_i - a)^T H (p_i - a) \leq 1, \text{ for all } i = 1, \dots, n_p.$$

The volume of the ellipsoid is proportional to  $1/\det(H)$ . Therefore, writing  $H = LL^T$  ( $L$  lower-triangular with non-negative diagonal), the problem of minimizing the area reduces to maximize  $\sum_{i=1}^{n_d} \log(\ell_{ii})$  and the constraints are

$$(p_i - a)^T LL^T (p_i - a) \leq 1, \text{ for all } i = 1, \dots, n_p$$

with the obvious bounds  $\ell_{ii} > 0$  for all  $i = 1, \dots, n_d$ .

In this way, the problem results in an interesting variable-dimension Nonlinear Programming test.

However, the interesting case is when there does not exist a close formula for the volume of the sets in the class  $\mathcal{C}$ . Assume, for example, that the elements of  $\mathcal{C}$  are intersections of two ellipsoids (Figure 5). Therefore, the constraints of the NLP problem will be:

$$(p_i - a_1)^T H_1 (p_i - a_1) \leq 1, (p_i - a_2)^T H_2 (p_i - a_2) \leq 1,$$

for all  $i = 1, \dots, n_p$ .

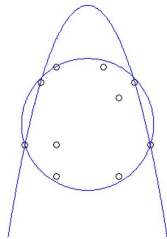


Figure 5: Minimal-area intersection of 2 ellipses enclosing 10 given points.

The unknowns of the problem are the coefficients of  $H_1$  and  $H_2$  and the centers  $a_1, a_2 \in \mathbb{R}^{n_d}$ . Writing  $H_1$  and  $H_2$  in factorized form we can guarantee that these matrices are positive semidefinite.

The complication is that the volume of the intersection of two ellipsoids cannot be computed analytically, except in particular cases. Therefore, it must be computed approximately using some simulation procedure. As a consequence, the derivatives are not available and the problem can only be solved using a derivative-free technique. Because of these

facts, this type of problems generate many interesting tests for Derivative-Free Nonlinear Programming. Of course, there exist many realistic problems where derivatives are not available, but they usually depend on complicate models, black-box routines and data availability. Therefore, derivative-free methods are many times tested using problems in which derivatives are available (but not used) and noise is artificially simulated. In the objective functions of the geometric problems presented here, the derivatives are really not-available but testing and coding is not hard. Moreover, since simulation is usually costly, the problems also evoke the fact that, in real-life non-derivative problems, functions coming from models are hard to compute. Finally, volumes computed by simulation may be naturally noisy and smoothness is not guaranteed [10, 20].

Figure 6 shows a case where the class  $\mathcal{C}$  is formed by the intersections between an ellipse and a rectangle.

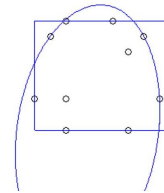


Figure 6: Minimal-area intersection of rectangle and ellipse containing 10 given points.

In Figure 7 a slightly different situation is shown. The sets in  $\mathcal{C}$  are unions of two ellipses. Observe that constraints of the form

$$(x - a_1)^T H_1 (x - a_1) \leq 1 \text{ or } (x - a_2)^T H_2 (x - a_2) \leq 1$$

may be written as

$$\min\{(x - a_1)^T H_1 (x - a_1), (x - a_2)^T H_2 (x - a_2)\} \leq 1.$$

## 6. Sentinels

An informal description of the concept of sentinels [4] is the following: Suppose that  $A$  and  $B$  are two identical “solid bodies”. For example,  $A$  and  $B$  may be two rectangles with the same dimensions without a fixed position. Let  $P_1, \dots, P_{n_A}$  be a set of points in

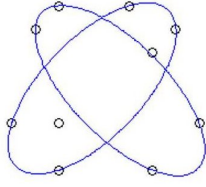


Figure 7: Minimal-area union of two ellipses enclosing 10 given points.

$A$  and  $Q_1, \dots, Q_{n_B}$  be a set of points in  $B$ . We say that  $P_1, \dots, P_{n_A}$  and  $Q_1, \dots, Q_{n_B}$  are *Sentinels* if the fact that  $A$  and  $B$  “overlap” implies that a point  $P_i$  overlaps  $B$ , or a point  $Q_i$  overlaps  $A$  [4]. Therefore, if the objects  $A$  and  $B$  are not superimposed, no sentinel of  $A$  invades  $B$  and no sentinel of  $B$  invades  $A$ .

In general, the determination of the minimum number of sentinels related to solid bodies is not simple. In Figure 8 the minimal set of sentinels corresponding to identical rectangles is shown [4].

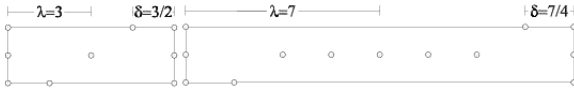


Figure 8: Minimal sets of rectangle sentinels.

Assume now that one wants to place  $n_p$  identical convex polyhedra in a region  $\mathcal{R} \subset \mathbb{R}^{n_d}$  defined by convex inequalities  $g_1(x) \leq 0, \dots, g_p(x) \leq 0$ . The fact that the polyhedra must be contained in  $\mathcal{R}$  is represented saying that each polyhedron vertex satisfies  $g_i(x) \leq 0, i = 1, \dots, p$ .

Assume, moreover, that a suitable arrangement of sentinels is known. The non-overlapping condition says that, for each sentinel  $s$  of the polyhedron  $P_i$ , and for all  $j \neq i$ , one has  $s$  is not interior to  $P_j$ .

The unknowns of the problem are the translation vector and the rotation angles that, for each polyhedron  $P_i$ , define its position within  $\mathcal{R}$ . Vertices and sentinels are, in fact, functions of translations and angles.

Several different objective functions can be proposed. If the region  $\mathcal{R}$  is not fixed, but depends of some variable parameters, we may minimize some geometrical quantity that also depends of these parameters (length, width, area, *etc.*). Figure 9 shows a case in which 30 rectangles were packed in the in-

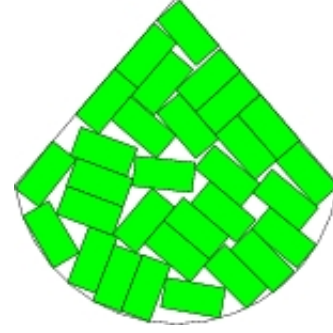


Figure 9: Packing rectangles by means of the Method of Sentinels.

tersection of a triangle and a circle using the method of sentinels [4].

## 7. Packing balls

Figure 10 shows a putative solution of the problem of finding the minimal-size tetrahedron that encloses 50 balls of radius 1, and 50 balls of radii 1.025, 1.050,  $\dots$ , 2.250. The edge of this tetrahedron is  $\approx 27.33$  [7]. The unknowns of the problem are the centers  $p_1, \dots, p_{n_{balls}}$  of the balls ( $n_{balls} = 100$  in Figure 10). The constraints that say that the balls do not overlap are:

$$\|p_i - p_j\|_2^2 \geq (r_i + r_j)^2 \text{ for all } i < j, \quad (2)$$

where  $r_k$  is the given radius of the ball  $k$ , for  $k = 1, \dots, n_{balls}$ .

The constraints that say that the balls are enclosed in the tetrahedron are represented in [7] as:

$$2\sqrt{2}p_i^x - 2\sqrt{6}p_i^y + 2p_i^z \leq \sqrt{6}L - 6r_i,$$

$$2\sqrt{2}p_i^x + 2\sqrt{6}p_i^y + 2p_i^z \leq \sqrt{6}L - 6r_i,$$

$$-2\sqrt{2}p_i^x + p_i^z + r_i \leq 0,$$

$$p_i^z \geq 0,$$

for all  $i = 1, n_{balls}$ , where the 3D coordinates of  $p_i$  are represented  $(p_i^x, p_i^y, p_i^z)$  and  $L$  is the edge size. The problem is to minimize  $L$  subject to these constraints and the constraints (2).

As in the other problems described in this article, there is no guarantee that the exhibited solution is a global one. The solution was obtained using a

“local” optimization algorithm coupled with a multistart procedure. The output examination shows that there are many local minimizers and, probably, non-minimum stationary points.

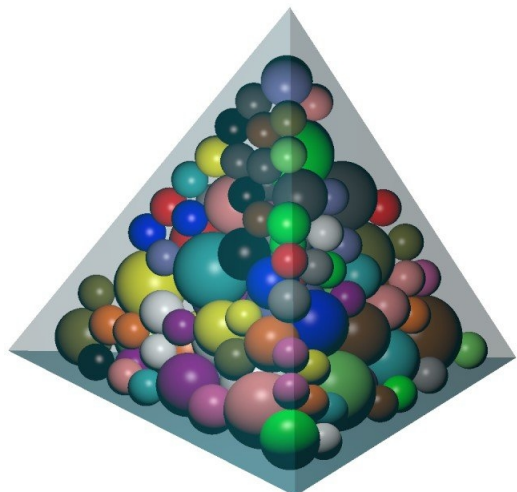


Figure 10: Minimal tetrahedron enclosing 100 given balls.

## 8. PACKMOL

PACKMOL is a computational package for placing molecules in a given domain in such a way that overlap is minimized [16] (<http://www.ime.unicamp.br/~martinez/ackmol>.) It is used in the context of Molecular Dynamics to find initial configurations for which the starting energy is moderate. Figure 11 shows typical solutions obtained by PACKMOL that have been successfully used in further Molecular Dynamics (MD) computations.

PACKMOL solves a real-life applied problem. Real data regarding its use depend on the particular MD application and, so, to use PACKMOL for regular NLP tests is not straightforward.

However, a simple geometric problem lies behind the PACKMOL philosophy. Assume that the molecules to be packed are  $M_1, \dots, M_{n_{mol}}$ . Each molecule  $M_i$  may be thought as a set of points (atoms) in  $\mathbb{R}^3$ , say,  $M_i = \{p_1^i, \dots, p_{n(i)}^i\}$ .

The unknowns of the problem are the rigid displacements  $D_1, \dots, D_{n_{mol}}$ . Under the rigid displacement  $D_i$ , the molecule  $M_i$  goes from its original position to  $D_i(M_i) = \{D_i(p_1^i), \dots, D_i(p_{n(i)}^i)\}$ . A

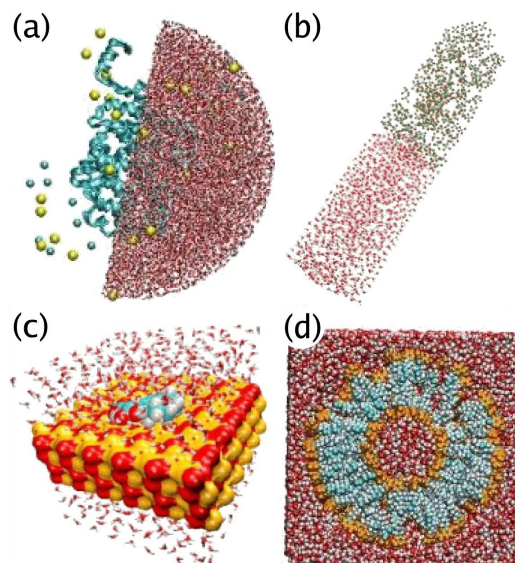


Figure 11: (a) Protein solvated by water and ions, containing 53 thousand atoms ( $\sim 100$  thousand variables) [17]. (b) Interface between water and carbon tetrachloride [18]. (c) Molecule in interface of water and zirconia [19]. (d) Double-layered vesicle with water inside and outside (70 thousands atoms).

rigid displacement is represented in PACKMOL by a translation vector and three rotation (Euler) angles. The displaced molecules must be contained in the selected set  $\mathcal{B}$ , according to the MD requirements.

Therefore, it may be thought that the original problem is to maximize the minimal distance between  $D_i(p_j^i)$  and  $D_\ell(p_k^\ell)$  ( $i \neq \ell$ ) subject to

$$D_i(p_j^i) \in \mathcal{B}, i = 1, \dots, n_{mol}, j = 1, \dots, n(i). \quad (3)$$

In other words, one wishes to maximize  $z$  subject to the constraints (3) and

$$z \leq \|D_i(p_j^i) - D_\ell(p_k^\ell)\|_2^2, i \neq \ell. \quad (4)$$

## 9. Solving the problems

Most problems mentioned in this article share a common characteristic. They possess, potentially, a large number of inequality constraints, most of which tend to be inactive at the solution. Efficient NLP algorithms for solving the problems should take into account this feature. On one hand, algebraic manipulation of this type of constraints or their derivatives is, generally, a waste of time. On the other hand, it

is convenient to avoid even the computation of constraints that are predictably inactive, without impairing the convergence properties of the algorithm. In many geometrically structured problems, the second objective is not hard to achieve: Inactive constraints are usually associated with the position of two points in the space. Briefly, if the points are distant, the corresponding constraint is inactive. Unfortunately, the evaluation of all pairwise distances may be very expensive. The remedy is to employ a previous classification of the points in boxes, in such a way that only points belonging to neighboring boxes may define constraints that deserve to be computed. In this way, roughly speaking, the computer time of evaluating all the constraint information ceases to be a quadratic function of the number of points and, under reasonable assumptions, is linear. This procedure, closely connected with the linked-cell strategies used in Molecular Dynamics [12], is used in the web-available version of PACK-MOL and turns out to be crucial for the practicality of the method.

The former considerations lead one to revisit the Powell-Hestenes-Rockafellar (PHR) version of the Augmented Lagrangian method [1, 8, 15, 21, 22]. PHR algorithms rely on the iterative approximate minimization of the Augmented Lagrangian  $L_\rho(x, \lambda, \mu)$  defined by

$$L_\rho(x, \lambda, \mu) = f(x) + \frac{\rho}{2} \left[ \left\| h(x) + \frac{\lambda}{\rho} \right\|_2^2 + \left\| \left( g(x) + \frac{\mu}{\rho} \right)_+ \right\|_2^2 \right]$$

on a simple set  $\Omega \subset \mathbb{R}^n$ . Details are given in [1], where the ALGENCAN method, that uses GENCAN [3] as box-constrained solver, is described and its theoretical properties are proved. If, at the current point  $x$ , one has that  $g_i(x) \leq -\mu_i/\rho$ , then no calculation related to  $g_i$  is needed. The absence of slack variables to complete the constraints  $g_i(x) \leq 0$  causes second derivative discontinuity, a cost that seems to be affordable in the problems under consideration.

Augmented Lagrangian methods are not believed to be competitive with modern Interior Point methods that use Sequential Quadratic Programming and — many times — filters [11] for many problems available in the literature on practical optimization. The

main reason is that their final speed of convergence is rather slow (generally linear instead of superlinear and quadratic). However, in problems with many inactive inequality constraints and problems with very poor KKT-matrix structure, the PHR Augmented Lagrangian method is an interesting alternative.

For generating the initial approximation, the polar representation of the sphere was employed. Taking  $n_{grid} \geq 3$ ,  $n_p$  points in the unitary sphere of  $\mathbb{R}^{n_d}$  were generated in the following way:

1. Compute  $2 \times n_{grid}$  equally spaced “angles”  $\varphi_1 \in [0, 2\pi]$ . For  $i = 2, \dots, n_d - 1$ , compute  $n_{grid}$  equally spaced angles in  $\varphi_i \in [-\pi/2 + \delta/2, \pi/2 - \delta/2]$ , where  $\delta = \pi/n_{grid}$ .
2. For defining the initial approximation, compute  $n_p$  points in the unitary sphere. Each point  $p^k$  is generated taking:

$$p_1^k = \cos(\varphi_1) \times \cos(\varphi_2) \times \dots \times \cos(\varphi_{n_d-1}),$$

$$p_i^k = \sin(\varphi_{i-1}) \times \cos(\varphi_i) \times \dots \times \cos(\varphi_{n_d-1}),$$

for  $i = 2, \dots, n_d - 1$ ,

$$p_{n_d}^k = \sin(\varphi_{n_d-1}),$$

for all the combinations of angles so far defined. Therefore,  $n_p = 2 \times n_{grid}^{n_d-1}$ . The initial approximation  $x^0$  was formed by  $p^1, \dots, p^{n_p}$  followed by the variable  $z = x_{n+1}^0$ . The initial  $z$  was taken as the maximum scalar product  $\langle p^i, p^j \rangle$  for  $i \neq j$ .

The selected problems are defined by  $n_d = 3$  and  $n_{grid} = 7, 8, 9$ . Therefore,  $n_p = 98, 128, 162$ .

The dimensional characteristics of the selected Hard-Spheres problems are:

- **Hard-Spheres (3, 98):**  $n_d = 3$ ,  $n_p = 98$ ,  $n$  without slacks: 295,  $n$  with slacks: 5048, number of equality constraints: 98, number of inequality constraints: 4753, total number of constraints: 4851.
- **Hard-Spheres (3, 128):**  $n_d = 3$ ,  $n_p = 128$ ,  $n$  without slacks: 385,  $n$  with slacks: 8513, number of equality constraints: 128, number of



inequality constraints: 8128, total number of constraints: 8256.

- **Hard-Spheres (3,162):**  $n_d = 3$ ,  $n_p = 162$ ,  $n$  without slacks: 487,  $n$  with slacks: 13528, number of equality constraints: 162, number of inequality constraints: 13041, total number of constraints: 13203.

The column “Infeasibility” of Table 1 reports the final sup-norm of constraint violations,  $f$  is the final objective function value and “ $k$ ” is the number of iterations.

Table 1 illustrates the efficiency of the PHR approach when the number of inequality constraints is large. These examples *should not* be used to make efficiency or reliability claims regarding problems where the many-inequalities characteristic is not present.

Problem	Infeasibility	Final $f$	$k$	Time
(3, 98)	3.5057E-10	9.3273E-01	8	8.17
(3, 128)	3.7860E-11	9.4825E-01	10	25.94
(3, 162)	3.7424E-11	9.5889E-01	10	40.15

Table 1: Three Hard-Spheres problems.

## 10. Conclusion

In “The Analytical Language of John Wilkins”, Jorge Luis Borges describes a Chinese Encyclopedia in which animals are divided into:

1. those that belong to the Emperor,
2. embalmed ones,
3. those that are trained,
4. suckling pigs,
5. mermaids,
6. fabulous ones,
7. stray dogs,
8. those included in the present classification,
9. those that tremble as if they were mad,
10. innumerable ones,
11. those drawn with a very fine camelhair brush,
12. others,
13. those that have just broken a flower vase,
14. those that from a long way off look like flies.

The obvious difficulties in classifying nonconvex optimization problems leads one to evoke Borges’ animal taxonomy. Many times, as practical optimizers, we discover ourselves talking about “problems belonging to the Cute Collection”, “problems with more than 10000 variables”, “nonlinear least-squares problems”, “problems coming from Chemical Engineering” and so on. A rigorous, complete, clear and useful classification seems to be impossible, even if one is restricted to “practical problems” (which, in fact, is also a dubious category). The “geometric problems” commented in this article are candidates to constitute a “class” of optimization problems, at least in the sense of Borges. As “the animals that have just broken a flower vase”, they have a number of features in common but none of these features appear exclusively in problems of this class. On the other hand, different problems in the geometric class may be very simple or extremely difficult and may differ in a good number of characteristics. Perhaps, the most useful denomination for this class is: Problems for which a nice picture can be drawn. Nevertheless, the presence of geometrical characteristics to which optimality can be associated represents a powerful stimulus for the development of algorithmic ideas.

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# Extracting Salient Features From Less Data via $\ell_1$ -Minimization

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MRI (magnetic resonance imaging) is a widely used medical imaging modality that creates an image from scanned data that are essentially the Fourier coefficients of this image. A typical abdominal scan may take around 90 minutes. Can we reduce this time to 30 minutes by using one third of the Fourier coefficients, while maintaining image quality? In this article, we hope to convince the reader that such reductions are achievable through a new and promising approach called *compressive sensing* (or compressed sensing). The main computational engine that drives compressive sensing is  $\ell_1$ -related minimization algorithms.

## 1. Introduction

Exploiting sparsity is a common task in computational sciences, as it is in signal processing. Recently, sparsity has been skillfully utilized to increase data acquisition capacity in a new approach called *compressive sensing*. Seminal contributions in this area include Candés and Tao [3] and Donoho [11]. In a nutshell, this technique encodes a sparse signal into a shorter “code” whose length is roughly proportional to the sparsity level rather than the length of the signal. The decoding process, on the other hand, involves solving an optimization problem. This is very different from the traditional paradigm where a full-length code is first acquired, then compressed, and the decoding process is relatively inexpensive. This paradigm shift can potentially bring great benefits to certain applications. However, solving large-scale optimization problems arising from compressive sensing poses real challenges.

### 1.1 A synthetic example

Let us try to acquire a sparse signal  $\bar{x} \in \mathbb{R}^n$  of length  $n = 200$  depicted in Figure 1(a). Let  $k = \|\bar{x}\|_0$  be the number of nonzeros in  $\bar{x}$ , which is 10. First,  $\bar{x}$  is encoded into a “compressed code”  $b = R\bar{x} \in \mathbb{R}^m$ ,

$m < n$ , by a linear transform  $R$ . Typically in signal acquisition practice, such encoding is not calculated on a computer but obtained by certain physical or digital means. Notice that since  $\bar{x}$  is “unknown” at this time,  $R$  can only be constructed independently (non-adaptively) of  $\bar{x}$ . In this synthetic example, we let  $R \in \mathbb{R}^{m \times n}$  be formed from a subset of  $m = 80$  rows of the  $n$ -dimensional discrete cosine transform (DCT) matrix  $\Phi$ . The number  $m$  is called the *sample size*.  $\Phi\bar{x}$  and  $R\bar{x}$  are depicted in Figures 1(b) and (c) where those in  $\Phi\bar{x}$  but not in  $R\bar{x}$ , *i.e.*, the missing measurements, are replaced with zeros in (c). After the compressed code  $b = R\bar{x}$  is acquired by a sensor and becomes available, we need to *decode* it to recover the original signal. That is where optimization enters the picture. Although the linear equations  $Rx = b$  have an infinite number of solutions because  $m < n$ , one may use the fact that the number of nonzeros in  $\bar{x}$ ,  $\|\bar{x}\|_0$ , is small and try to recover  $\bar{x}$  as the solution to the  $\ell_0$ -problem:

$$\min_{x \in \mathbb{R}^n} \{\|x\|_0 : Ax = b\} \quad (1)$$

for  $A = R$ , where the “ $\ell_0$ -norm” of  $x$  is the number of nonzeros in  $x$ . The solution of (1) will be  $\bar{x}$  unless there exists another solution to  $Rx = b$  that is equally sparse or sparser than  $\bar{x}$  (which does not happen under favorable conditions; see next section). However, the  $\ell_0$ -problem (1) is combinatorial and generally NP-hard [26]. A much more tractable alternative is the  $\ell_1$ -problem (also called *basis pursuit*):

$$\min_{x \in \mathbb{R}^n} \{\|x\|_1 : Ax = b\}, \quad (2)$$

which is a convex program that always has a solution whenever  $Ax = b$  is consistent. As we will show in Section 2, problem (2) yields the same solution as the  $\ell_0$ -problem under some mild conditions,

From a different perspective, this is also an example of missing data recovery [35]. Given a portion of data  $b$  (Figure 1(c)) that is known, one can recover the complete data  $f$  (Figure 1(b)) by exploiting the sparsity of  $\bar{x}$  representing  $f$  under a basis  $\Phi$ , *i.e.*,  $\Phi\bar{x} = f$ . Specifically, solving (2), for  $A$  equal to the sub-matrix of  $\Phi$  corresponding to  $b$ , gives the optimal solution  $x_{\text{opt}} = \bar{x}$  so that the original signal is reconstructed as  $f = \Phi x_{\text{opt}}$  (Figure 1(e)).

Ideally, we would like to take the smallest number of measurements possible, that is,  $m = \bar{k} \equiv \|\bar{x}\|_0$ .

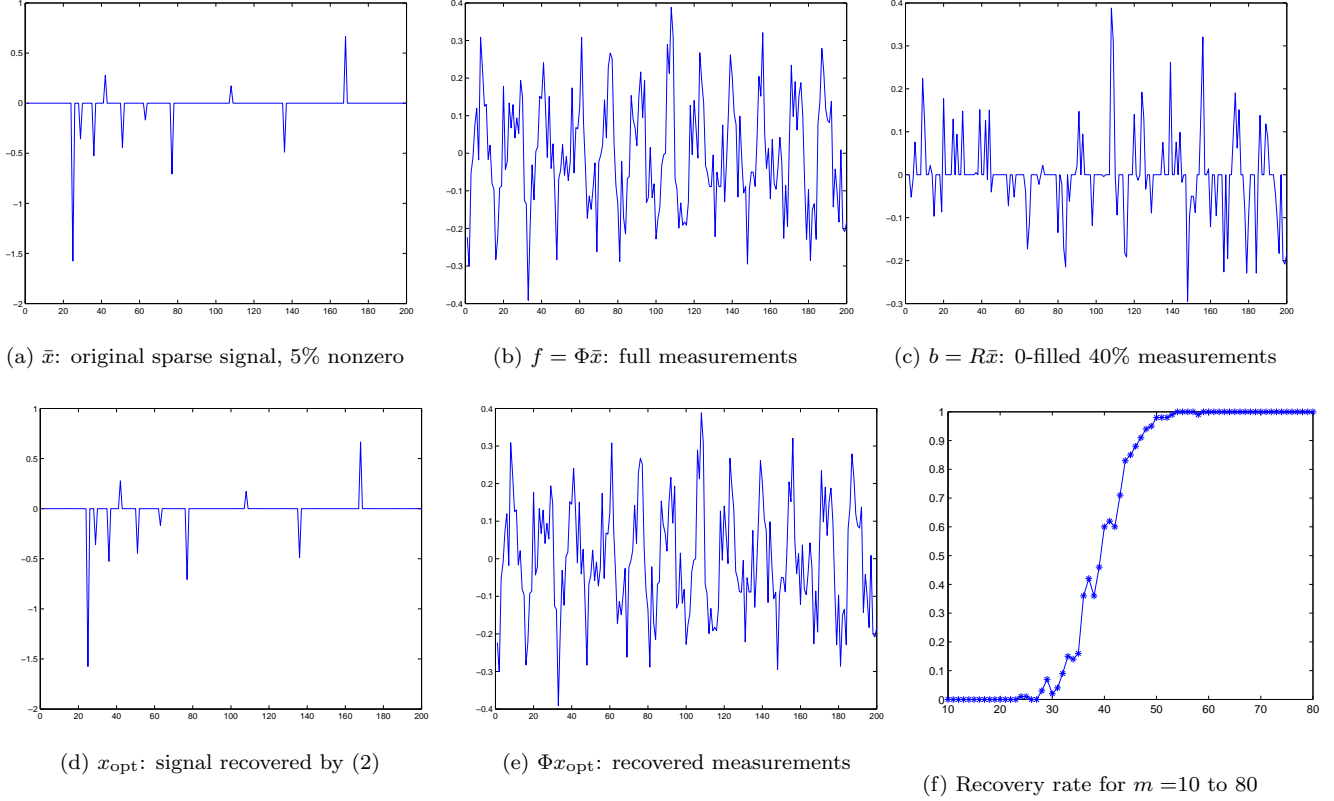


Figure 1: Signal recovery from under-sampled measurement.  $\Phi$  is a discrete cosine transform.

However, we must pay a price for not knowing the locations of the nonzeros in  $\bar{x}$  (there are “ $n$  choose  $\bar{k}$ ” possibilities!). It was shown in [3, 4, 30] that, when  $R$  is a certain random matrix, then by solving (2) for  $m = O(\bar{k} \log(n/\bar{k}))$  one can recover  $\bar{x}$  with high probability. Though larger than  $\bar{k}$ , such an  $m$  can be asymptotically much smaller than  $n$  when  $\bar{k} \ll n$ .

To illustrate this point, we performed similar calculations depicted above for  $m = 10, 11, \dots, 80$ , each with 100 repetitions of randomly chosen  $m$  measurements. The percentages of successful recovery for all  $m$  are plotted in Figure 1(e), which shows that it is generally safe to have  $m > 6\|\bar{x}\|_0 = 60$  for this combination of  $\|\bar{x}\|_0$ ,  $n$ , and  $A$ .

## 1.2 Hidden sparsity

If compressive sensing were only applicable to spatially and temporally sparse signals, it would have few applications. Most images, for example, are not sparse in the pixel domain, but rather have sparse representations in either the Fourier (spectral) or Wavelet (spectral-scale) domain. Let the vector  $\bar{u}$

represent such an image. A compression algorithm (*e.g.* JPEG2000) would find an invertible matrix  $\Phi$  (*e.g.*, a wavelet basis) such that the vector  $\bar{x} = \Phi \bar{u}$  has a relatively small number of large-magnitude components. Let  $\tilde{x}$  be the sparse vector formed by taking only the large-magnitude components of  $\bar{x}$ . Then,  $\bar{u}$  can be accurately approximated by  $\Phi^{-1}\tilde{x}$ . This is not surprising because the useful information in most images is relatively sparse compared to pixel values. Since  $\Phi \bar{u}$  is sparse, one can recover  $\bar{u}$  from  $b = R\bar{u}$  by solving

$$\min_u \{\|\Phi u\|_1 : Ru = b\}, \quad (3)$$

which is equivalent to solving (2) with  $x = \Phi u$  and  $A = R\Phi^{-1}$ .

Like images, many signals by their nature are sparse in certain domains. The principle of compressive sensing is that such a signal can be recovered from a relatively small number of measurements provided its sparsity is appropriately exploited. However, a good sparse representation for a given signal is not always obvious. Recently, some progress has been made on signals arising from low-light imaging,

medical imaging (MRI and CT), infra-red sensing, bio-sensing, radar signal processing, multi-sensor networks and distributive sensing, and analog-to-information conversion. The interested reader can visit the Rice compressive sensing website [6] for a list of recent papers.

## 2. When are the $\ell_0$ - and $\ell_1$ -problems equivalent?

We give an informal proof of the fact that whenever  $A$  is random,  $\bar{x}$  is sufficiently sparse and  $b = A\bar{x}$ , then with high probability  $\bar{x}$  will solve the “basis pursuit” problem (2). Following the proof in [34], we will use a classic result developed by Kashin [21], and Garnaev and Gluskin [19].

### 2.1 A sufficient condition for recovery

We first derive a sufficient condition for  $\bar{x}$  to be the unique solution of (2) assuming that  $A \in \mathbb{R}^{m \times n}$  has rank  $m$  and  $m < n$ . Let  $\bar{x}$  satisfy  $A\bar{x} = b$  and denote the null space of  $A$  by  $\text{Null}(A)$ . Since

$$\{x : Ax = b\} \equiv \{\bar{x} + v : v \in \text{Null}(A)\},$$

$\bar{x}$  uniquely solves (2) if and only if

$$\|\bar{x} + v\|_1 > \|\bar{x}\|_1, \quad \forall v \in \text{Null}(A) \setminus \{0\}. \quad (4)$$

Let  $S$  be the support of  $\bar{x}$  and  $Z$  be its complement, *i.e.*,

$$S = \{i : \bar{x}_i \neq 0\}, \quad Z = \{i : \bar{x}_i = 0\},$$

and  $v_S$  be the sub-vector of  $v$  corresponding to the index set  $S$  (we apply similar notation for other vectors). Then we calculate

$$\begin{aligned} \|\bar{x} + v\|_1 &= \|\bar{x}_S + v_S\|_1 + \|0 + v_Z\|_1 \\ &= \|\bar{x}\|_1 + (\|v_Z\|_1 - \|v_S\|_1) + \\ &\quad (\|\bar{x}_S + v_S\|_1 - \|\bar{x}_S\|_1 + \|v_S\|_1), \end{aligned}$$

where in the right-hand side we have added and subtracted the terms  $\|\bar{x}\|_1$  and  $\|v_S\|_1$  (noting that  $\|\bar{x}\|_1 = \|\bar{x}_S\|_1$  given that  $\bar{x}_Z = 0$ ).

In the above identity, the last term in parentheses is nonnegative by the triangle inequality; hence,  $\|\bar{x} + v\|_1 > \|\bar{x}\|_1$  if  $\|v_Z\|_1 > \|v_S\|_1$ . Therefore, a sufficient

condition for  $\bar{x}$  to be the unique solution of (2) is that  $\|v_Z\|_1 > \|v_S\|_1$ , or equivalently  $\|v\|_1 > 2\|v_S\|_1$ , for all nonzero  $v$  in the null space of  $A$ . In view of the inequality

$$\|v_S\|_1 \leq \sqrt{|S|}\|v_S\|_2 \leq \sqrt{\|\bar{x}\|_0}\|v\|_2,$$

where we used the facts that (i) the length of  $v_S$  is  $|S|$  (the cardinality of the set  $S$ ) which equals  $\|\bar{x}\|_0$ , and (ii)  $v_S$  is a sub-vector of  $v$ , we derive another sufficient condition that  $\bar{x}$  uniquely solves (2) if

$$\sqrt{\|\bar{x}\|_0} < \frac{1}{2} \frac{\|v\|_1}{\|v\|_2}, \quad \forall v \in \text{Null}(A) \setminus \{0\}. \quad (5)$$

This condition requires nothing but sparsity of  $\bar{x}$  for it to solve (2) uniquely. This uniqueness implies that there can exist at most one vector  $\bar{x} \in \{x : Ax = b\}$  whose sparsity meets the condition (5); otherwise, it would not be the unique solution of (2). Such an  $\bar{x}$ , whenever it exists, must be the sparsest solution to  $Ax = b$ . In other words, *the  $\ell_1$ - and  $\ell_0$ -problems are equivalent* in the sense

$$\begin{aligned} \bar{x} &= \arg \min\{\|x\|_1 : Ax = b\} \\ &= \arg \min\{\|x\|_0 : Ax = b\}. \end{aligned} \quad (6)$$

The remaining question is how restrictive the condition (5) is? More precisely, how big can the bound on the right-hand side of (5) be? The answer will, of course, depend on the properties of matrix  $A$ .

### 2.2 Kashin-Garnaev-Gluskin result

We will make use of a classic result established in the late 1970’s and early 1980’s by Russian mathematicians. In our context, this result has to do with the ratio of the  $\ell_1$ -norm to the  $\ell_2$ -norm restricted to a subspace. We know that in the entire space  $\mathbb{R}^n$ , the ratio can vary from 1 to  $\sqrt{n}$ , namely,

$$1 \leq \frac{\|v\|_1}{\|v\|_2} \leq \sqrt{n}, \quad \forall v \in \mathbb{R}^n \setminus \{0\}.$$

Here we will only concern ourselves with the lower bound. Roughly, this ratio is small for sparse vectors that have many zero or near-zero elements. However, it turns out that in many subspaces this ratio can have much larger lower bounds than 1.

As an improvement to an earlier result by Kashin [21], Garnaev and Gluskin [19] established that for

any natural number  $p < n$ , there exist  $p$ -dimensional subspaces  $V_p \subset \mathbb{R}^n$  in which

$$\frac{\|v\|_1}{\|v\|_2} \geq \frac{C\sqrt{n-p}}{\sqrt{\log(n/(n-p))}}, \forall v \in V_p \setminus \{0\}, \quad (7)$$

where  $C$  is an absolute constant independent of the dimensions. In other words, these subspaces do not contain excessively sparse vectors. Moreover, such subspaces are abundant because *every  $p$ -dimensional subspace spanned by iid (independently identically distributed) random vectors of the standard Gaussian distribution will satisfy inequality (7) with high probability.* (This is an instance of a mathematical phenomenon commonly referred to as *concentration of measure*; see [25], for example.)

### 2.3 How sparse is enough?

If  $A$  is an  $m$  by  $n$  random matrix with iid standard-Gaussian entries, then it is known that the null space of  $A$  can be spanned by iid random vectors. In particular, vectors in the null space of  $A$  will satisfy, with high probability, the Garnaev and Gluskin inequality (7) for  $V_p = \text{Null}(A)$  and  $p = n - m$ . Combining the sufficient condition (5) with the Garnaev and Gluskin inequality (7), we have the result that for a random Gaussian matrix  $A$ ,  $\bar{x}$  will uniquely solve (2) with high probability whenever

$$\|\bar{x}\|_0 < \frac{C^2}{4} \frac{m}{\log(n/m)}. \quad (8)$$

(The constant  $C$  above is the same one from (7)). This result can be interpreted as follows. As long as the sparsity of a signal  $\bar{x}$  is less than a certain fraction of the number of random measurements  $m$ , where the value of the fraction only logarithmically deteriorates as the signal dimension  $n$  increases, with high probability this signal can be recovered from the random measurements by solving the basis-pursuit problem (2).

The sparsity bound given in (8) is the best order currently available, first established in [3] for Gaussian random matrices, which is a significant improvement upon previously existing bounds. The same order has been extended to some other random matrices such as Bernoulli matrices whose entries are  $\pm 1$  [4]. For certain partial orthonormal (for example, partial DCT) matrices, a slightly weaker bound

has been proved [30]. Moreover, an in-depth study on the constant in (8),  $C^2/4$ , can be found in [12].

## 3. Imaging and other applications

To demonstrate the potential benefit of compressive sensing in practical applications, let us simulate a compressed MRI (Magnetic Resonance Imaging) experiment using under-sampled measurements (see [24] for a more realistic work).

### 3.1 Compressed MRI simulation

First, we need an abridged overview of MRI — a non-invasive and safe medical imaging technique. In MRI, images are obtained in the form of the frequency response of tissues. First, a strong magnetic field and an RF (radio frequency) pulse are directed to a section of the anatomy, causing the protons in that area to be “excited”: they get aligned along the magnetic direction and spin with a certain frequency. Next, on turning off the RF pulse, the protons return to their natural, rather chaotic, state while releasing RF signals that are captured by external coils in the form of phases and magnitudes at selected frequencies. In other words, the image of spatial energy (or density), denoted by  $\bar{u}$ , is constructed from data acquired in the frequency domain, the so-called  $k$ -space. Roughly, at a given resolution, a complete set of sampled frequencies is  $f = \mathcal{F}\bar{u}$  where  $\mathcal{F}$  is a discrete Fourier transform. Therefore, an image can be constructed through a Fourier inversion  $\bar{u} = \mathcal{F}^{-1}f$ .

An MRI scan can be a long and uncomfortable process. For example, a patient must repeatedly hold his/her breath during an abdominal scan while strictly immobilized throughout the process, which can last 1 to 2 hours. Potentially, compressive sensing can help construct  $\bar{u}$  with a much smaller number of sample frequencies. This would mean that the MRI scan duration could be significantly reduced.

MR images often have sparse representations under some wavelet transform  $\Phi$ . By solving  $\min_u \{\|\Phi u\|_1 : Ru = b\}$  or its variants, we can obtain a given image  $\bar{u}$  from an under-sampled frequency set  $b = R\bar{u}$ , where  $R$  represents a partial discrete Fourier transform.

Let us simulate this approach to see how much compressive sensing could help. Figure 2(a) depicts

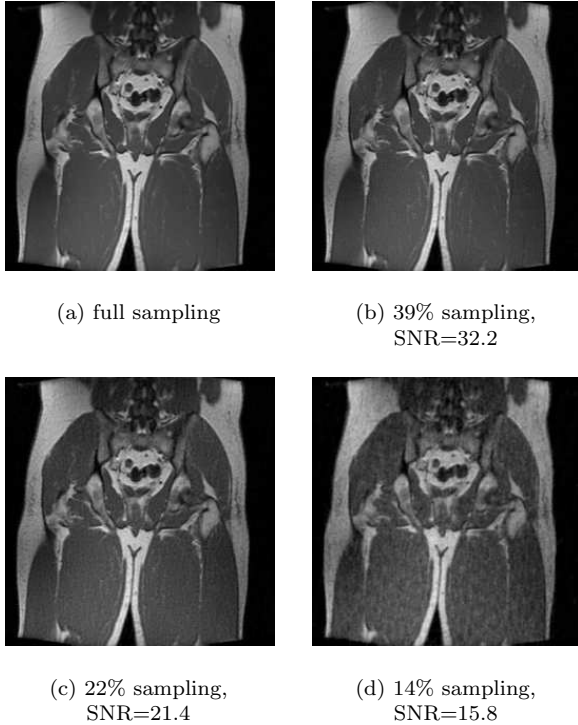


Figure 2: (a) Original Image; (b)-(d) Pelvis MR images recovered from incomplete measurements using the wavelet-based model (9) (where the higher the SNR (signal-noise ratio) is, the better the image quality is).

a clean  $256 \times 256$  pelvis MR image that is our  $\bar{u}$ . We tried the use of 39%, 22%, and 14% of its Fourier coefficients to reconstruct the image, simulating three different levels of under-sampling. Since in MR imaging, one has some freedom in selecting sample frequencies (however, practical constraints do exist), the reconstruction results were obtained by random sampling with a strong bias towards lower frequencies. The images in Figure 2(b)–(d) were obtained by solving the model

$$\min_u \|\Phi u\|_1 + \frac{\mu}{2} \|Ru - b\|^2, \quad (9)$$

with a Haar-wavelet transform  $\Phi$  and a large  $\mu$ . From a visual examination, it seems that using one third of measurements, if properly chosen, would be quite sufficient for obtaining a high-quality image for this case.

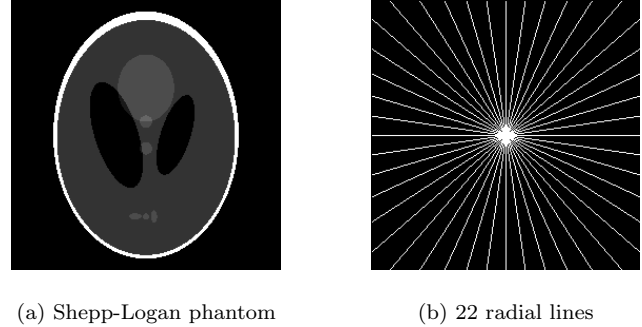


Figure 3: Fourier samples taken at the frequencies on the 22 radial lines (b) are sufficient to exactly recover Shepp-Logan phantom (a) using total variation.

### 3.2 Total variation

Many natural images possess a “blocky” structure. For such images, minimizing the *total variation* (cf. [36]) yields a better image quality [31]. For a 2D digital image  $u$ , the total variation of  $u$ ,  $TV(u)$ , is defined as the sum of Euclidean norms of local finite differences, *i.e.*,  $TV(u) \equiv \sum_{i,j} \|(Du)_{ij}\|_2$ , where  $(Du)_{ij}$  represents a first-order finite difference vector of  $u$  at pixel  $(i, j)$ . Since  $TV(u)$  is the  $\ell_1$ -norm of “gradient magnitude”, minimizing  $TV(u)$  tends to yield a solution with sparse finite differences, namely an image with constant-intensity blocks. Therefore, total-variation regularization has been widely used in image processing tasks such as noise removal, deblurring, edge detection, *etc.*. A similar argument can be used to justify the use of higher-order finite differences to regularize images of appropriate characteristics.

In [2], Candés and Romberg demonstrated that, by minimizing the total-variation, the Shepp-Logan phantom in Figure 3(a) can be almost exactly recovered from Fourier samples taken on 22 radial lines depicted in Figure 3(b).

Since finite difference operators are not invertible, minimizing total variation cannot be directly transformed into a problem of the form (2). This poses a major algorithmic challenge.

### 3.3 Broad applications

There are other potential applications of compressive sensing besides MRI, especially in areas where signal acquisitions are relatively expensive and time-consuming. For example, an infrared sensor is over a

hundred times more expensive than an image sensor of the same resolution in a consumer digital camera. In a CT (computed tomography) scan, a series of two-dimensional X-rays are used to construct a three-dimensional image, but a long exposure to the radiation from X-rays can be dangerous. In wireless sensor networks for collecting physical or environmental measurements, a large number of spatially distributed sensors acquire and transmit a deluge of data, relying on low capacity batteries. In all of these examples, physical hardware capacities are being stressed, and improving their sensing resolution or speed is expensive. Compressive sensing offers an invaluable alternative to expensive physical improvements by using much cheaper computing power after data collection. Some recent explorations and applications of compressive sensing can be found at the Rice compressive sensing website [6].

## 4. Algorithms

Let us turn to optimization, the ultimate tool for obtaining a sparse signal (or its sparse representation) from under-sampled measurements.

### 4.1 Formulations and challenges

Let  $J(x)$  be a convex, sparsity-promoting function, such as the  $\ell_1$ -norm or the total variation. To recover a sparse signal representation  $\bar{x}$  from measurements  $b \approx A\bar{x}$ , we can either solve

$$\min_x \{J(x) : Ax = b\}, \quad (10)$$

when  $b$  is relatively accurate, or solve

$$\min_x \{J(x) : H(Ax, b) \leq \epsilon\} \quad (11)$$

when  $b$  is more noisy, where  $H$  is a measure of the closeness of  $Ax$  to  $b$ . For an appropriate penalty parameter  $\mu$  (which can be found by a noise-statistics computation, cross validation, or simply trial and error), (11) is equivalent to

$$\min_x J(x) + \mu H(Ax, b) \quad (12)$$

for some  $\mu > 0$ . The most common choices of  $J$  and  $H$  are, respectively,  $J(x) = \|x\|_1$  and  $H(Ax, b) = \frac{1}{2}\|Ax - b\|_2^2$ . In Statistics, minimizing this  $H$  subject to  $\|x\|_1 \leq \delta$  is the so-called LASSO problem.

More generally, the regularization term  $J(x)$  can be a mixture of multiple terms representing multiple features of a sparse solution. For example, a signal may possess a piece-wise constant feature and have a sparse representation under a certain transform  $\Phi$  at the same time. In this case, we may use a mixed regularization term:

$$J(x) = TV(u) + \lambda \|\Phi x\|_1.$$

Similarly, the fidelity-measure function  $H(x)$  could also consist of multiple terms.

All these problems are non-smooth convex optimization problems that can be easily transformed into smooth problems with convex constraints. However, algorithmic challenges arise from the facts that (i) real-world application problems are invariably large-scale (an image of  $1024 \times 1024$  resolution leads to over a million variables); (ii) the data matrices involved are generally dense; and (iii) real-time or near real-time processing is often required (as in MRI). For these problems, conventional algorithms requiring matrix factorizations are generally not effective or even applicable.

On the other hand, when  $A$  is a partial transform matrix, fast matrix-vector multiplications are often available. Moreover, the sparsity in solutions presents unusual opportunities to achieve relatively fast convergence with first-order methods (*i.e.*, methods of gradient-descent type that do not solve linear systems during iterations). These features make the development of efficient optimization algorithms for compressive sensing applications an interesting research area.

### 4.2 Some recent algorithms

We mention a few algorithms recently developed for solving large-scale compressive sensing problems, fully realizing that any such list would be unavoidably incomplete. In addition to those briefly reviewed below, there are many other algorithms based upon ideas such as minimizing a non-convex  $\ell_p$ -“quasi-norm” for  $p < 1$ , iteratively weighted least squares, group testing, homotopy methods in statistics, combinatorial methods, and  $\ell_1$ -Bregman iterations. We again refer the reader to the Rice CS resource website [6] for more comprehensive lists of algorithmic papers and software.



Orthogonal Matching Pursuit (OMP) based methods (e.g., [32, 13, 7]) do not solve (2) *per se*, but use an iterative greedy approach to identify nonzero (or large-magnitude) components of  $x$  so that the residual  $b - Ax$  is minimized in some sense while keeping other components of  $x$  at zero. The recent algorithm StOMP [13] is a good representative of such greedy algorithms that can perform well on problems with highly sparse solutions and noiseless measurements.

A recent code called  $\ell_1\text{-}\ell_s$  [22] is based on an interior-point algorithm that uses a preconditioned conjugate gradient (PCG) method to approximately solve linear systems in a truncated-Newton framework. The authors exploit the structure of the Hessian to construct their preconditioner. Their computational results show that about a hundred PCG steps are sufficient for obtaining accurate MRI images in the compressive sensing framework. Though generally slower than first-order methods, this algorithm may offer a certain advantage on problems of less sparsity where first-order methods could potentially encounter slow convergence.

The recent method GPSR [18], which stands for gradient projection for sparse reconstruction, reformulates the unconstrained version (12) of (2) into a quadratic program with nonnegativity constraints and applies a projected gradient algorithm, with optional Barzilai-Borwein steps and a non-monotone line search. Although motivated from very different viewpoints, this algorithm has a certain similarity with shrinkage methods introduced below; however, their performance can be quite different on some problems.

SPGL1 [33] is a recent code for solving a sequence of problems of the form

$$\min_x \|Ax - b\|, \quad \text{s.t. } \|x\|_1 \leq \lambda, \quad (13)$$

for  $\lambda = \lambda_1, \lambda_2, \dots, \lambda_j = \bar{\lambda}$  until reaching the desired value  $\bar{\lambda}$ . The choice of  $\lambda$  is based on a root finding algorithm (e.g., Newton’s method) using two results: (i) the curve formed by the minimizers  $x_{\text{opt}}(\lambda)$  is convex and continuously differentiable in  $\lambda$ , (ii) the dual solution of (13) gives the gradient of the curve at  $\lambda$ .

Recently, a general method was proposed in [27] for minimizing  $J(x) + H(x)$ , where  $J$  is non-smooth,  $H$  is smooth, and both are convex. It is required that  $J$  be “simple” so that there exists a closed-form

solution to minimizing  $J$  plus some auxiliary functions. The  $\ell_1$ -norm is such a “simple” function since the problem  $\min_x \lambda \|x\|_1 + \frac{1}{2} \|x - y\|_2^2$  has the closed-form solution  $\text{shrink}(y, \lambda)$ , which is defined in (16) below. When  $H$  has Lipschitz continuous gradients, the objective value in this method converges at a rate  $O(k^{-2})$ , where  $k$  is the iteration number. This result shows that in general, minimizing the sum of  $J$  and  $H$  is not harder than minimizing the smooth function  $H$  alone as long as  $J$  is “simple”.

A widely used method for solving  $\ell_1$ -minimization problems of the form

$$\min_u \mu \|u\|_1 + H(u), \quad (14)$$

for a convex and differentiable  $H$ , is an iterative procedure based on shrinkage (also called soft thresholding; see (16) below). In the context of solving (14) with a quadratic  $H$ , this method was independently proposed and analyzed in [17, 28, 10, 1], and then further studied or extended in [14, 15, 9, 5, 20, 8]. It turns out that this algorithm can be directly derived from the classic forward-backward operator splitting technique (c.f. [23]). The basic shrinkage algorithm can be written as the fixed-point iteration: for  $i = 1, \dots, n$ ,

$$u_i^{k+1} = \text{shrink}((u^k - \tau \nabla H(u^k))_i, \mu \tau), \quad (15)$$

where  $\tau > 0$  serves as a step-length for gradient descent (which may vary with  $k$ ) and

$$\text{shrink}(t, \alpha) = t - \text{Proj}_{[-\alpha, \alpha]}(t) \quad (16)$$

for any  $t \in \mathbb{R}$  and  $\alpha > 0$ . It is easy to see that the larger  $\mu$  is, the larger the allowable distance between  $u^{k+1}$  and  $u^k$ .

A new result in [20] is the finite convergence of the support and the signs of  $u^k$  under a non-degeneracy condition. That is,  $\text{sign}(u^k) \equiv \text{sign}(u_{\text{opt}})$  (assuming  $\text{sign}(0) = 0$ ) for all  $k \geq K$ , where  $u_{\text{opt}}$  denotes the solution of (14) (however, an estimate or bound for  $K$  is still unknown). It was also proved in [20] that the rate of convergence is  $q$ -linear under suitable conditions on  $\tau$  and  $H$ , and the rate depends on the condition of a sub-Hessian, rather than the entire Hessian, of  $H$  at  $u_{\text{opt}}$ . These results provide explanations why sparsity in solutions can help accelerate convergence of first-order methods.

Various modifications and enhancements have been proposed to improve the efficiency of the basic iteration (15), including [16, 18]. In our view, the basic iteration (15) would not be practically effective without a continuation (or path-following) strategy [20, 33] in which a gradually decreasing sequence of  $\mu$ -values is used to guide the iterates towards the final optimal solution. In [20], the performance of a fixed-point continuation (FPC) algorithm was compared with those of StOMP [13], GPSR [18] and  $\ell_1$ - $\ell_s$  [22].

In addition, a general block-coordinate gradient descent method for linearly constrained separable problems [29] can be applied to solving (14).

## 5. Concluding remarks

Compressive sensing is a new, application-driven, interdisciplinary area where optimization can have a great impact. Given the diversity of applications, successful algorithms should be able to take full advantage of problem structure. We have just seen the beginning of activities in this direction.

Taking advantage of sparsity has always been one of the central tasks in computational algorithms. However, it is fair to say that most previous efforts have been concentrated on sparsity in problem data rather than sparsity in solutions. How to optimally exploit solution sparsity certainly deserves closer examinations in algorithmic studies.

Noise and errors naturally appear in measurements in practical applications. A good algorithm for compressive sensing should be robust with respect to noise and errors under normal conditions. Comprehensive and in-depth research in this direction has yet to be conducted.

Unlike for most other problems, algorithm designers for compressive sensing have some freedom in selecting problem data. For example, which measurement matrix should we use for a given problem, a random Gaussian or a partial DCT matrix? Which frequencies should we sample in MRI? This interaction between problem data, sparse solution and algorithms presents a rich and unique set of research opportunities. Moreover, if data are acquired over a period of time, can we develop a “warm-start” algorithm that produces approximate solutions whose accuracy progressively improves with the increase in

available measurements?

The past few years have seen a burst of activities using  $\ell_1$ -related optimization in areas such as statistics, machine learning, signal processing, imaging, and computer vision. While the gradient-descent method is probably the most well-known and widely used tool, researchers in these areas have developed rich analytical results and efficient computational tools for solving various  $\ell_1$ -related optimization problems. Historically, research in optimization has been stimulated by the demand of engineering applications, and subsequently contributed to the practice of these applications. We believe that today we are witnessing the same phenomenon repeating itself in the area of  $\ell_1$ -related optimization.

## Acknowledgments

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## Bulletin

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Note that SIAM associates are entitled to the reduced conference fee as the members of SIMAI.

Renato Spigler  
Organizing Committee SIMAI2008

### 1. Obituary

We regret to announce the death of I. I. Dikin on February 28th from a heart attack. Dikin proposed the affine-scaling interior-point method for linear and quadratic programming as a student of L. V. Kantorovitch in 1967, and proved its convergence in a 1974 paper. The Dikin ellipsoid is a well-known construct in interior-point methods for linear and convex programming. Dikin worked at the Siberian Energy Institute, applying mathematical programming and interior-point methods to energy problems.

Roman Polyak and Michael J. Todd

### 2. Event Announcements

#### SIMAI2008

An international conference organized in cooperation with SIAM

May 15–19, 2008

Rome, Italy

<http://www.simai.eu>

The Italian Society for Applied and Industrial Mathematics (SIMAI) will held its 9th Congress in Rome, Italy from September 15th to 19th. This international event takes place every two years. This time it is being organized in cooperation with SIAM.

The hosting environment will be the beautiful downtown of Rome. The main (invited) speakers will be Antonio Ambrosetti (SISSA, Trieste), Douglas N. Arnold (University of Minnesota, USA), Nicola Bellomo (Politecnico di Torino), Giovanni Ciccotti (Università di Roma “La Sapienza”), Nicholas J. Higham (University of Manchester, UK), and Alfio Quarteroni (Politecnico di Milano & EPFL Lausanne).

Besides, there will be a number of minisymposia (usually several dozens), round tables, prizes to young mathematicians, and interactions with industrial representatives. More information (in progress) can be found at the SIMAI site.

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## Chairman's Column

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The first thing I would like to do is to thank the outgoing Chair, Kurt Anstreicher for all his efforts over the past 3 years on our behalf, and also to acknowledge the enormous contributions made by the vice chair, Bob Vanderbei, the program director, Sven Leyffer and the secretary/treasurer Kees Roos. I have quickly learned of the efforts these folk have put in on our behalf, and I remain grateful for the very healthy shape in which we find the activity group.

I am excited about seeing you again at the Optimization meeting in Boston. In addition to hearing about your innovative research ideas, I note that the Optimization prize will be awarded, and that SIAM promises 'refreshments' for those who attend the Business Meeting! In addition to our standard business, Steve Vavasis, our new program director, and I are soliciting ideas for possible locations of the next SIAM Optimization meeting in 2011.

Tom McCormick, our new secretary, is transferring the web site to a standard location on SIAM's web server, and we hope to collect new content to enhance this service. Please send suggestions directly to him. The new web site is at: <http://siags.siam.org/siagopt/>

If you have ideas on new directions or changes that you believe would benefit our SIAG, please let me, or Yinyu Ye, our new vice chair, know about them. While these matters may provoke lots of discussion, are there mechanisms that will help us forge closer ties to our sister organizations within INFORMS, MPS and AMS? Suggestions are always welcomed, but particularly if they come with a promise of volunteered help.

Finally, I would like to thank Luis Vicente for all his excellent work in preparing our newsletters for the past 6 years. He has made this a vital part of our activities and I am pleased to announce that

Sven Leyffer has agreed to take over this important task and serve for the next three years as Newsletter editor.

**Michael C. Ferris**, SIAG/OPT Chair  
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## Comments from the Editor

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This issue of SIAM/Optimization Views-and-News is one of my favorites, even though it contains two unrelated articles. This is simply because these articles are extremely interesting, novel and very nicely written. It is my duty, but also my pleasure, to thank José Mario Martínez, Wotao Yin and Yin Zhang for graciously accepting my invitation and, of course, for having put so much time and effort into the preparation of their articles.

I have been editing this bulletin since the Fall of 2003. I started when Henry Wolkowicz was chair, continued under the directorship of Kurt Anstreicher, and finish up working with Michael Ferris. I always felt from them and from all the other SIAG officers a strong and resilient support. While I am pleased to relinquish this post after such a long tenure as editor, I am happy to leave the bulletin in the capable hands of Sven Leyffer.

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