SIAG/OPT Views-and-News

A Forum for the SIAM Activity Group on Optimization

Volume 15 Number 2 October 2004

Contents

Sums of Squares Optimization via Semidefinite Programming

Introduction by the Guest Editor

Semidefinite programming (SDP) has been called "linear programming for the year 2000", and has been one of the most popular research areas in optimization in the last ten years. One of the exiting new applications of SDP is to obtain convex relaxations of polynomial optimization problems.

This issue of SIAG/OPT Views-and-News includes two expository papers on the underlying methodology, as well as a case study in control theory. In particular:

- Lasserre shows how to obtain a hierarchy of SDP relaxations of polynomial optimization problems via the theory of moment matrices;
- Parrilo explains the dual approach, which relies on the theory of positive polynomials;
- Henrion and Ghildiyal show how to apply the SDP methodology to a problem in control theory, namely how to design a robust H_∞ controller of fixed order for the inverted pendulum.

Etienne de Klerk, October 11, 2004.

SDP-Relaxations for Polynomial Programming

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Abstract: We briefly describe a recent technique to solve polynomial programming problems, which uses algebraic representation theorems for polynomials, positive on a compact set.

1. Introduction

In Nonlinear Programming (NLP), one handles functions that are not necessarily polynomials, and, therefore, basic concepts, tools and techniques are usually borrowed from real and functional analysis as well as standard linear algebra, rather than commutative algebra and algebraic geometry. As underlined in Wright [27], Linear Programming (LP) was an exception, and until the *interior points* revolution, LP could be taught from a purely algebraic and geometric point of view, and completely separated from NLP. In counterpart to the large spectrum of NLP problems with say, continuous functions, and with the exception of *convex* problems, one is usually concerned with only *local optimality* conditions and *local search* algorithms.

If one is interested in the global optimum, then observe that the NLP problem $\mathbb{P} \to f^* := \min_{x\in\mathbb{K}} f(x)$ has the two following equivalent formulations

$$f^* = \min_{\mu} \left\{ \int_{\mathbb{R}^n} f\, d\mu \mid \quad \mu(\mathbb{K}) = 1;\, \mu(\mathbb{R}^n \setminus \mathbb{K}) = 0 \right\} \tag{1}$$

where the minimization is over the set of measures μ on $\mathbb{R}^n$, and

$$f^* = \max_{\lambda\in\mathbb{R}} \{\lambda \mid \quad f - \lambda \geq 0 \text{ on } \mathbb{K}\}. \tag{2}$$

Of course, the LP (1) and its dual (2) are just a rephrasing of $\mathbb{P}$ and are useless in general, unless one knows either (a) how to evaluate $\int f\, d\mu$ efficiently and characterize measures with support on $\mathbb{K}$, or (b), characterize functions that are nonnegative on $\mathbb{K}$.

This is precisely the case for the class of NLP problems

$$\mathbb{P} \quad \to f^* = \min\{f(x) \mid \quad g_i(x) \geq 0,\, i = 1, \ldots, m\} \tag{3}$$

with *polynomial* constraints and criterion, that we call *Polynomial Programming* (PP) problems. That is, $\{f, \{g_i\}_{i=1}^m\} \subset \mathbb{R}[x_1, \ldots, x_n]$ and $\mathbb{K} \subset \mathbb{R}^n$ is the semi-algebraic set

$$\mathbb{K} := \{x \in \mathbb{R}^n \mid \quad g_i(x) \geq 0, \qquad i = 1, \ldots, m\}. \tag{4}$$

(A semi-algebraic set of $\mathbb{R}^n$ is a set defined by polynomial inequalities.) Using recent results from (real) algebraic geometry, one may provide a new paradigm for PP problems, based on a sequence of *convex SDP-relaxations* (of increasing size), whose associated monotone sequence of optimal values converges to f^* under fairly weak assumptions. This new framework is indeed realistic because of recent progress in *Semidefinite Programming* (SDP) which makes those SDP-relaxations practically implementable. The reader should keep in mind that most interesting PP problems are NP-hard. However, the above mentioned SDP-relaxations provide a monotone nondecreasing sequence of lower bounds and very often, f^* is actually obtained early in the sequence. Despite this appealing feature, and in view of the present status of SDP (still at its early stage), one is currently limited in the size of PP problems that can be handled so far.

In our mind, this new approach is particularly appealing for the three main reasons below, to be developed later.

(a) *Generality:* The class of PP problems is very large and encompasses many interesting applications. For instance, nonconvex quadratic problems and Mixed-Integer Programming (MIP) problems are particular PP problems. The feasible set $\mathbb{K}$ neither needs to be connected nor continuous, as it can be any semi-algebraic set represented by finitely many polynomial inequalities.

(b) *Global optimality conditions:* Putinar, Jacobi and Prestel's refinement of Schmüdgen's Positivstellensatz theorem on the representation of polynomials positive on a compact set, is nothing less than the *global optimality conditions* analogue for PP of the celebrated Karush-Kuhn-Tucker local optimality conditions for NLP. In addition, the algebraic point of view (2) of positive polynomials has a dual functional analysis point of view (1), namely the theory of *moments*.

(c) *The tool:* SDP is the appropriate tool that fits a certain representation of positive polynomials (algebraic point of view) and its dual theory of moments (functional analysis) mentioned in (b). It is indeed a remarkable fact that the primal SDP-relaxations have a natural interpretation in the theory of moments, whereas the dual SDP-relaxations also have a natural interpretation in the representation theory (algebraic point of view).

This approach initiated by Shor [23] and later by Lasserre [11, 12], Nesterov [16], Parrilo [17], has been implemented in the two software packages GloptiPoly [17] and SOSTOOLS [18], both available on the net, and results on a sample of problems are indeed very encouraging and promising. Namely, for most problems of the sample in [17], the optimal value f^* is actually obtained (up to the precision machine) at a SDP-relaxation of low order.

Finally, we also briefly describe LP-relaxations of a similar algebraic flavor, initiated in the pioneering work of Sherali and Adams [21, 22] for 0-1 programs, and recently extended to general PP problems in [14] by invoking a representation result of Krivine [9, 10].

2. Notation and preliminaries

Given any two real-valued symmetric matrices A, B let $\langle A, B\rangle$ denote the usual scalar product trace(AB) and let $A \succeq B$ (resp. $A \succ B$) stand for $A-B$ positive semidefinite (resp. $A-B$ positive definite). Let

$$[1, x_1, x_2, \ldots x_n, x_1^2, x_1x_2, \ldots, x_n^2, \ldots] \tag{5}$$

be the canonical basis for the real polynomials, and let $\mathcal{A}_r \subset \mathbb{R}[x] := \mathbb{R}[x_1, \ldots, x_n]$ be the subspace of real polynomials of degree at most r; denote $s(r)$ its dimension. Therefore, a polynomial $f \in \mathcal{A}_r$ is written

$$x \mapsto f(x) = \sum_{\alpha\in\mathbb{N}^n} f_\alpha x^\alpha = \sum_\alpha f_\alpha x_1^{\alpha_1}\cdots x_n^{\alpha_n},$$

$x \in \mathbb{R}^n$, in the basis (5), for finitely many coefficients $\{f_\alpha\}$ such that $|\alpha| = \sum_{i=1}^n \alpha_i \leq r$. The polynomial $f \in \mathcal{A}_r$ is also identified with its vector of coefficients $f \in \mathbb{R}^{s(r)}$ in the canonical basis (5).

2.1 Moment matrix

Given a sequence $y = \{y_\alpha\}$, the *moment* matrix $M_r(y) \in \mathbb{R}^{s(r)\times s(r)}$ with rows and columns indexed in (5), satisfies

$$[M_r(y)(1,j) = y_\alpha \text{ and } M_r(y)(i,1) = y_\beta] \implies M_r(y)(i,j) = y_{\alpha+\beta}.$$

For instance, with $n = 2$, $r = 2$, and $y = \{1, y_{10}, y_{01}, y_{20}, y_{11}, y_{02}\}$, we have

$$M_2(y) = \begin{bmatrix} 1 & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{bmatrix}.$$

The matrix $M_r(y)$ defines a bilinear form $\langle .,. \rangle_y$ on $\mathcal{A}_r$, by

$$\langle q, v\rangle_y := \langle q, M_r(y)v\rangle, \quad q, v \in \mathcal{A}_r,$$

and if y has a *representing* measure μ_y (*i.e.*, $y_\alpha = \int_{\mathbb{R}^n} x^\alpha \, d\mu_y$ for all $\alpha \in \mathbb{N}^n$), then

$$\langle q, M_r(y)q\rangle = \int_{\mathbb{R}^n} q(x)^2 \, \mu_y(dx) \geq 0, \tag{6}$$

so that $M_r(y) \succeq 0$.

2.2 Localizing matrix

Given a polynomial $\theta \in \mathbb{R}[x]$, with coefficient vector θ, the *localizing* matrix $M_r(\theta y) \in \mathbb{R}^{s(r)\times s(r)}$ associated with θ, is given by

$$M_r(\theta y)(i,j) = \sum_{\alpha\in\mathbb{N}^n} \theta_\alpha y_{\beta+\alpha} \quad \text{if} \quad M_r(y)(i,j) = y_\beta. \tag{7}$$

For instance with $n = 2$, $r = 1$, and $x \mapsto \theta(x) := a - x_1^2 - x_2^2$,

$$M_1(\theta y) = \begin{bmatrix} a - y_{20} - y_{02}, & ay_{10} - y_{30} - y_{12}, & ay_{01} - y_{21} - y_{03} \\ ay_{10} - y_{30} - y_{12}, & ay_{20} - y_{40} - y_{22}, & ay_{11} - y_{31} - y_{13} \\ ay_{01} - y_{21} - y_{03}, & ay_{11} - y_{31} - y_{13}, & ay_{02} - y_{22} - y_{04} \end{bmatrix}.$$

If y is has a representing measure μ_y, then

$$\langle q, M_r(\theta y)q\rangle = \int_{\mathbb{R}^n} \theta(x)q(x)^2 \, \mu_y(dx), \tag{8}$$

for every polynomial $q \in \mathcal{A}_r$. Therefore, $M_r(\theta y) \succeq 0$ whenever μ_y has its support contained in the set $\mathbb{K}_\theta = \{x \in \mathbb{R}^n \mid \theta(x) \geq 0\}$ (*i.e.*, $\mu(\mathbb{R}^n \setminus \mathbb{K}_\theta) = 0$).

2.3 Putinar-Jacobi-Prestel representation theorem

The $\mathbb{K}$-moment problem identifies those sequences $y = \{y_\alpha\}$ that have a representing measure whose support is contained in a semi-algebraic set $\mathbb{K}$. In duality with the theory of moments is the theory of representation of positive polynomials, which dates back to Hilbert's 17th problem. For details and recent results, the interested reader is referred to Berg [1], Curto and Fialkow [3], Jacobi and Prestel [7], Putinar [19], Simon [24], Schmüdgen [20], and the many references therein. Recall that a polynomial $f \in \mathbb{R}[x]$ is a *sum of squares* (s.o.s.) if it can be written $f = \sum_{j\in J} f_j^2$ for some finite family $\{f_j\}_{j\in J} \subset \mathbb{R}[x]$.

Assumption 1 *Let $\{g_i\}_{i=1}^m \subset \mathbb{R}[x]$ and let $\mathbb{K} \subset \mathbb{R}^n$ be as in (4). The set $\mathbb{K}$ is compact and there exists some $u \in \mathbb{R}[x]$ such that the level set $\{x \in \mathbb{R}^n \,|\, u(x) \geq 0\}$ is compact, and u can be written as $u = u_0 + \sum_{i=1}^m u_i g_i$, where $u_i \in \mathbb{R}[x]$ is s.o.s. for every $i = 0, 1, \ldots, m$.*

Note that Assumption 1 is not very restrictive. For instance, it holds whenever $\mathbb{K}$ is a convex polytope, or the level set $\{x \in \mathbb{R}^n \,|g_i(x) \geq 0\}$ is compact for some $i = 1, \ldots, m$. In addition, if one knows in advance that a global minimizer of $\mathbb{P}$ is contained in a ball of radius M, then for Assumption 1 to hold, it suffices to include the additional quadratic constraint $M^2 - \|x\|^2 \geq 0$ in the definition (4) of $\mathbb{K}$. If Assumption 1 holds then one has the following important representation:

$$[\, f \in \mathbb{R}[x], \quad f \,>\, 0 \text{ on } \mathbb{K}\,] \Rightarrow f \,=\, f_0 + \sum_{i=1}^m f_i\, g_i, \tag{9}$$

where $f_i \in \mathbb{R}[x]$ is s.o.s. for every $i = 0, 1, \ldots, m$ (see *e.g.* Putinar [19]).

This representation (9) is a refinement of a more general representation theorem called Schmüdgen's Positivstellensatz [20], in which products of the g_i's would also appear in (9). Note that the representation (9) of f is an obvious *certificate of positivity* of f on $\mathbb{K}$. In addition, checking (9) with an *a priori* bound on the degree of the f_j's, reduces to solving a SDP.

The *dual* (functional analysis) analogue of (the algebraic) (9) is as follows: Let $\mathbb{K}$ be as in (4), $y = \{y_\alpha\}$ be an infinite sequence indexed in the basis (5). Then under Assumption 1, there exists a representing measure μ with support contained in $\mathbb{K}$, if and only if

$$M_r(y) \succeq 0, \; M_r(g_i y) \succeq 0, \tag{10}$$

$i = 1, \ldots, m; r = 0, 1, \ldots$ (again see Putinar [19]). Checking (10) for a single r reduces to solving a SDP.

3. SDP-relaxations for polynomial programming

Let $f \in \mathbb{R}[x]$, $\mathbb{K}$ be as in (4), and consider the (global) optimization problem PP problem $\mathbb{P}$ in (3).

For every $r \in \mathbb{N}$, write the moment and localizing matrices $M_r(y)$ and $M_r(g_i y)$ in the respective forms $\sum_\alpha y_\alpha B_\alpha$, and $\sum_\alpha y_\alpha C_\alpha^i$, for appropriate symmetric matrices $\{B_\alpha, C_\alpha^i\}$. Depending on its parity, let $\deg g_i = 2v_i$ or $2v_i - 1$ for all $i = 1, \ldots, m$, whereas $\deg f = 2v_0$ or $2v_0 - 1$, and set $i_0 := \max_{j \in \{0,\ldots,m\}} v_j$. For $r \geq i_0$ define the SDP problem

$$\mathbb{Q}_r \left\{ \begin{array}{lll} & \displaystyle\min_y \sum_\alpha f_\alpha y_\alpha & \\ \text{s.t.} & M_r(y) & \succeq \; 0 \\ & M_{r-v_k}(g_k y) & \succeq \; 0, \quad k = 1, \ldots, m \\ & y_0 & = \; 1 \end{array} \right. \tag{11}$$

(which is a SDP-relaxation of $\mathbb{P}$), whose dual SDP problem reads

$$\mathbb{Q}_r^* \left\{ \begin{array}{llll} & \displaystyle\max_\lambda \lambda & & \\ \text{s.t.} & \displaystyle\langle X, B_\alpha\rangle + \sum_{k=1}^m \langle Z_k, C_\alpha^k\rangle & = & f_\alpha, \\ & & & \forall\, \alpha \neq 0 \\ & \displaystyle\langle X, B_0\rangle + \sum_{k=1}^m \langle Z_k, C_0^k\rangle + \lambda & = & f_0 \\ & \forall k \in \{1, \ldots, m\} \quad X, Z_k & \succeq & 0, \end{array} \right. \tag{12}$$

Introducing the constant polynomial $g_0 \equiv 1$, every SDP-relaxation $\mathbb{Q}_r^*$ can be rephrased as

$$\sup \mathbb{Q}_r^* \;=\; \max_{\lambda \in \mathbb{R}} \left\{ \lambda \,|\, f - \lambda \,=\, \sum_{j=0}^m f_j\, g_j, \; f_j \text{ is s.o.s.} \right. \\ \left. \text{and } \deg f_j g_j \leq 2r, \quad \forall j = 0, \ldots, m \right\},$$

whereas every SDP-relaxation $\mathbb{Q}_r$ aims at finding a measure μ that minimizes $\int_{\mathbb{R}^n} f\, d\mu$ under the constraints that $\int_{\mathbb{R}^n} q^2 g_k\, d\mu \geq 0$ for all $q \in \mathcal{A}_{r-v_k}$, $k = 1, \ldots, m$.

3.1 Convergence

Let $\inf \mathbb{Q}_r, \sup \mathbb{Q}_r^*$ be the respective optimal values of the SDP $\mathbb{Q}_r$ and $\mathbb{Q}_r^*$ (writing min and max if the optimal value is attained). Then $\sup \mathbb{Q}_r^* \leq \inf \mathbb{Q}_r \leq f^*$ for all $r \geq i_0$, and under Assumption 1,

(a) $\inf \mathbb{Q}_r \uparrow f^*$ as $r \to \infty$.

(b) In addition, if $\mathbb{K}$ has nonempty interior, then $\sup \mathbb{Q}_r^* = \max \mathbb{Q}_r^* = \inf \mathbb{Q}_r$ for all $r \geq i_0$.

(c) Remember that $f - f^*$ is only nonnegative (and not strictly positive) on $\mathbb{K}$. But if $f - f^*$ has the representation (9) then for some $r_0 \in \mathbb{N}$, $\sup \mathbb{Q}_r^* = \max \mathbb{Q}_r^* = \min \mathbb{Q}_r = f^*$ for all $r \geq r_0$. That is, the SDP-relaxations $\mathbb{Q}_r$ are *exact* for all $r \geq r_0$.

For a detailed proof of (a)-(c) and more details, the interested reader is referred to Lasserre [11]. Finally, implemented in GloptiPoly [17], there is a sufficient (rank) condition to detect whether $\inf \mathbb{Q}_r = f^*$ at some given relaxation $\mathbb{Q}_r$, and if so, a procedure to retrieve a global minimizer $x^* \in \mathbb{K}$ from an optimal solution y^* of $\mathbb{Q}_r$.

3.2 Global optimality conditions

Observe that when the representation (9) holds for the polynomial $f - f^*$, nonnegative (but not strictly positive) on $\mathbb{K}$, then (9) is nothing less than the *global optimality conditions* analogue for PP of the Karush-Kuhn-Tucker (KKT) local optimality conditions for NLP. Indeed, let $x^* \in \mathbb{K}$ be a global minimizer, and let $f - f^*$ have the representation (9), i.e.,

$$f - f^* \;=\; f_0^* + \sum_{j=1}^m f_j^* g_j, \tag{13}$$

for some s.o.s. polynomials $\{f_j^*\}_{j=0}^m \subset \mathbb{R}[x]$. Then setting $\lambda_j := f_j^*(x^*) \geq 0$ for all $j = 1, \ldots, m$,

(i) $\lambda_j g_j(x^*) = 0$ for all $j = 1, \ldots, m$ (complementary slackness).

(ii) $\nabla f(x^*) = \sum_{j=1}^n \lambda_j \nabla g_j(x^*)$, with $\lambda_j = f_j^*(x^*) \geq 0$.

That is, $(x^*, \lambda) \in \mathbb{R}^n \times \mathbb{R}_+^m$ is a KKT pair. Conversely, if $(x^*, \lambda^*) \in \mathbb{R}^n \times \mathbb{R}_+^m$ is a KKT pair, and the gradients $\{\nabla g_j(x^*)\}$ are linearly independent, then $f_j^*(x^*) = \lambda_j^*$ for all $j = 1, \ldots, m$.

So (13) is the nonconvex *global optimality condition* analogue of the KKT optimality condition in the *convex* case, which states that

$$f - f^* - \sum_{j=1}^m \lambda_j^* g_j \geq 0,$$

and in which the *scalar* nonnegative multipliers $\{\lambda_j^*\}$ are now replaced with s.o.s. polynomials $\{f_j^*\}$.

3.3 Additional remarks

(i) As shown in the sample of problems solved with the software GloptiPoly [17], most of the time, the optimal value f^* is obtained (up to numerical machine precision) at a SDP-relaxation $\mathbb{Q}_r$ for some small r.

(ii) Mixed integer programming MIP problems as well as many discrete optimization problems are PP problems. In particular, for (nonlinear) $0-1$ programs (and more generally nonlinear integer programs) finite convergence is guaranteed (see, *e.g.* Lasserre [12]). For instance, for the MAXCUT problem (minimize a quadratic form on $\{-1,1\}^n$), the exact optimal value was almost always found at the second relaxation ($f^* = \min \mathbb{Q}_2$), wehereas the $0.878 \min \mathbb{Q}_1 \geq f^*$ guarantee follows from Goemans and Williamson [4] (for MAXCUT with nonnegative weights).

(iii) Finally, let $\mathbb{K}$ be as in (4), and let $\rho^* = \min_{x \in \mathbb{K}} f(x)/q(x)$, where $f, q \in \mathbb{R}[x]$ are relatively prime, and q does not change sign on $\mathbb{K}$. (If q would change sign on $\mathbb{K}$ then $\rho^* = -\infty$.) Then minimizing the rational function f/q on $\mathbb{K}$ is equivalent to solving

$$\min_{\mu} \left\{ \int_{\mathbb{R}^n} f \, d\mu \mid \quad \int q \, d\mu = 1; \quad \mu(\mathbb{R}^n \setminus \mathbb{K}) = 0 \right\},$$

from which converging SDP-relaxations in the spirit of the $\mathbb{Q}_i$'s can be built up easily (see de Klerk and Jibetean [8] for a dual point of view). Note that with $q \equiv 1$ one retrieves the PP problem (3).

3.4 LP-relaxations of $\mathbb{P}$

Suppose that in (4), the polynomials $\{0, 1, \{g_i\}_{i=1}^m\}$ *generate* the $\mathbb{R}$-algebra $\mathbb{R}[x_1, \ldots, x_n]$, that is, $\mathbb{R}[x_1, \ldots, x_n] = \mathbb{R}[g_1, \ldots, g_m]$. Assume also that the g_j's are normalized so that $0 \leq g_j \leq 1$ on $\mathbb{K}$, for all $j = 1, \ldots, m$. Then one may also define a sequence of LP-relaxations of $\mathbb{P}$, whose associated sequence of optimal values also converges to f^*. They are based on an algebraic representation theorem of Krivine [9, 10] (see also Vasilescu [26], and for a polytope $\mathbb{K}$, Cassier [2] and Handelman [5]), which states that

$$[\, f \;>\; 0 \text{ on } \mathbb{K}\,] \;\Rightarrow\; f = \sum_{\alpha, \beta \in \mathbb{N}^m} c_{\alpha\beta} \prod_{i=1}^m g_i^{\alpha_i} \prod_{j=1}^m (1 - g_j)^{\beta_j} \tag{14}$$

for finitely many *nonnegative* scalar coefficients $\{c_{\alpha\beta}\}$.

If one fixes an upper bound r on $|\alpha+\beta| = \sum_i(\alpha_i+\beta_i)$ in (14), then identifying the two polynomials on both sides of the equality in (14), yields a (finite) system of *linear* equalities on the nonnegative coefficients $\{c_{\alpha\beta}\}$. Therefore, the corresponding LP-relaxation $\mathbb{L}_r^*$ is the linear program

$$\max_{\lambda, c_{\alpha\beta}} \left\{ \lambda \,|\, f-\lambda = \sum_{\alpha,\beta\in\mathbb{N}^n} c_{\alpha\beta} \prod_{i=1}^m g_i^{\alpha_i} \prod_{j=1}^m (1-g_j)^{\beta_j}; \right. \\ \left. \{c_{\alpha\beta}\geq 0;\ |\alpha+\beta|\leq r\right\},$$

which, for obvious reasons, is the LP-analogue of the SDP-relaxation $\mathbb{Q}_r^*$. Similarly, the dual LP-relaxation $\mathbb{L}_r$, the analogue of the SDP-relaxation $\mathbb{Q}_r$, can also be interpreted in the theory of moments. Indeed, stating that a measure μ has its support on $\mathbb{K}$, can be done via countably many *linear conditions* on its moments $\{y_\alpha\}$. These conditions generalize to semi-algebraic sets the well-known linear Hausdorff-Bernstein conditions for the interval $[0,1]$, and one has proved in Lasserre [13, 14] that $\max \mathbb{L}_r^* = \min \mathbb{L}_r \uparrow f^*$ as $r\to\infty$. Finally, LP-relaxations in the spirit of the above have been defined in the pioneering work of Sherali and Adams [21] for $0-1$ nonlinear programs, in which they prove finite convergence, as for the SDP-relaxations. See also extensions in Sherali and Adams [22].

However, despite LP solvers can handle large size problems (in contrast to present SDP solvers), these LP-relaxations suffer some serious drawbacks analyzed in Lasserre [13, 14]. See also Laurent [15] for a comparison of SDP and LP-relaxations for $0-1$ programs.

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Sum of Squares Programs and Polynomial Inequalities

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1. Introduction

Consider a given system of polynomial equations and inequalities, for instance:

$$\begin{aligned} f_1(x_1,x_2) &:= \quad x_1^2 + x_2^2 - 1 = 0, \\ g_1(x_1,x_2) &:= \quad 3\,x_2 - x_1^3 - 2 \geq 0, \\ g_2(x_1,x_2) &:= \quad x_1 - 8\,x_2^3 \geq 0. \end{aligned} \tag{1}$$

How can one find real solutions (x_1, x_2)? How to *prove* that they do not exist? And if the solution set is nonempty, how to optimize a polynomial function over this set?

Until a few years ago, the default answer to these and similar questions would have been that the possible nonconvexity of the feasible set and/or objective function precludes any kind of analytic global results. Even today, the methods of choice for most practitioners would probably employ mostly local techniques (Newton's and its variations), possibly complemented by a systematic search using deterministic or stochastic exploration of the solution space, interval analysis or branch and bound.

However, very recently there have been renewed hopes for the efficient solution of specific instances of this kind of problems. The main reason is the appearance of methods that combine in a very interesting fashion ideas from *real algebraic geometry* and *convex optimization* [27, 30, 21]. As we will see, these methods are based on the intimate links between sum of squares decompositions for multivariate polynomials and semidefinite programming (SDP).

In this note we outline the essential elements of this new research approach as introduced in [30, 32], and provide pointers to the literature. The centerpieces will be the following two facts about multivariate polynomials and systems of polynomials inequalities:

> Sum of squares decompositions can be computed using semidefinite programming.

The search for infeasibility certificates is a convex problem. For bounded degree, it is an SDP.

In the rest of this note, we define the basic ideas needed to make the assertions above precise, and explain the relationship with earlier techniques. For this, we will introduce sum of squares polynomials and the notion of *sum of squares programs*. We then explain how to use them to provide infeasibility certificates for systems of polynomial inequalities, finally putting it all together via the surprising connections with optimization.

On a related but different note, we mention a growing body of work also aimed at the integration of ideas from algebra and optimization, but centered instead on integer programming and toric ideals; see for instance [7, 42, 3] and the volume [1] as starting points.

2. Sums of squares and SOS programs

Our notation is mostly standard. The *monomial* x^α associated to the n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ has the form $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, where $\alpha_i \in \mathbb{N}_0$. The *degree* of a monomial x^α is the nonnegative integer $\sum_{i=1}^n \alpha_i$. A *polynomial* is a finite linear combination of monomials $\sum_{\alpha\in S} c_\alpha x^\alpha$, where the coefficients c_α are real. If all the monomials have the same degree d, we will call the polynomial *homogeneous* of degree d. We denote the ring of multivariate polynomials with real coefficients in the indeterminates $\{x_1, \ldots, x_n\}$ as $\mathbb{R}[x]$.

A multivariate polynomial is a *sum of squares* (SOS) if it can be written as a sum of squares of other polynomials, i.e.,

$$p(x) = \sum_i q_i^2(x), \quad q_i(x) \in \mathbb{R}[x].$$

If $p(x)$ is SOS then clearly $p(x) \geq 0$ for all x. In general, SOS decompositions are not unique.

Example 1 *The polynomial $p(x_1, x_2) = x_1^2 - x_1x_2^2 + x_2^4 + 1$ is SOS. Among infinite others, it has the decompositions:*

$$\begin{aligned} p(x_1, x_2) &= \frac{3}{4}(x_1 - x_2^2)^2 + \frac{1}{4}(x_1 + x_2^2)^2 + 1 \\ &= \frac{1}{9}(3 - x_2^2)^2 + \frac{2}{3}x_2^2 + \\ &\quad + \frac{1}{288}(9x_1 - 16x_2^2)^2 + \frac{23}{32}x_1^2. \end{aligned}$$

The sum of squares condition is a quite natural sufficient test for polynomial nonnegativity. Its rich mathematical structure has been analyzed in detail in the past, notably by Reznick and his coauthors [6, 38], but until very recently the computational implications have not been fully explored. In the last few years there have been some very interesting new developments surrounding sums of squares, where several independent approaches have produced a wide array of results linking foundational questions in algebra with computational possibilities arising from convex optimization. Most of them employ semidefinite programming (SDP) as the essential computational tool. For completeness, we present in the next paragraph a brief summary of SDP.

Semidefinite programming SDP is a broad generalization of linear programming (LP), to the case of symmetric matrices. Denoting by $\mathcal{S}^n$ the space of $n \times n$ symmetric matrices, the standard SDP primal-dual formulation is:

$$\min_X \; C \bullet X \quad \text{s.t.} \begin{cases} A_i \bullet X = b_i, & i = 1, \ldots, m \\ X \succeq 0 \end{cases}$$

$$\max_y \; b^T y, \qquad \text{s.t.} \; \sum_{i=1}^m A_i y_i \preceq C, \tag{2}$$

where $A_i, C, X \in \mathcal{S}^n$ and $b, y \in \mathbb{R}^m$. The matrix inequalities are to be interpreted in the partial order induced by the positive semidefinite cone, *i.e.*, $X \succeq Y$ means that $X - Y$ is a positive semidefinite matrix. Since its appearance almost a decade ago (related ideas, such as eigenvalue optimization, have been around for decades) there has been a true "revolution" in computational methods, supported by an astonishing variety of applications. By now there are several excellent introductions to SDP; among them we mention the well-known work of Vandenberghe and Boyd [44] as a wonderful survey of the basic theory and initial applications, and the handbook [45] for a comprehensive treatment of the many aspects of the subject. Other survey works, covering different complementary aspects are the early work by Alizadeh [2], Goemans [15], as well as the more recent ones due to Todd [43], De Klerk [9] and Laurent and Rendl [25].

From SDP to SOS The main object of interest in semidefinite programming is

> *quadratic* forms, that are *positive semidefinite.*

When attempting to generalize this construction to homogeneous polynomials of higher degree, an unsurmountable difficulty that appears is the fact that deciding nonnegativity for quartic or higher degree forms is an NP-hard problem. Therefore, a computational tractable replacement for this is the following:

> *even degree* polynomials, that are *sums of squares.*

Sum of squares programs can then be defined as optimization problems over affine families of polynomials, subject to SOS contraints. Like SDPs, there are several possible equivalent descriptions. We choose below a free variables formulation, to highlight the analogy with the standard SDP dual form discussed above.

Definition 1 *A sum of squares program has the form*

$$\begin{array}{rl} \max_y & b_1 y_1 + \cdots + b_m y_m \\ s.t. & P_i(x,y) \ \textit{are SOS}, \quad i = 1, \dots, p, \end{array}$$

where $P_i(x,y) := C_i(x) + A_{i1}(x)y_1 + \cdots + A_{im}(x)y_m$*, and the* C_i, A_{ij} *are given polynomials in the variables* x_i*.*

SOS programs are very useful, since they directly operate with polynomials as their basic objects, thus providing a quite natural modelling formulation for many problems. Among others, examples for this are the search for Lyapunov functions for nonlinear systems [30, 28], probability inequalities [4], as well as the relaxations in [30, 21] discussed below.

Interestingly enough, despite their apparently greater generality, sum of squares programs are in fact *equivalent* to SDPs. On the one hand, by choosing the polynomials $C_i(x), A_{ij}(x)$ to be quadratic forms, we recover standard SDP. On the other hand, as we will see in the next section, it is possible to exactly embed every SOS program into a larger SDP. Nevertheless, the rich algebraic structure of SOS programs will allow us a much deeper understanding of their special properties, as well as enable customized, more efficient algorithms for their solution [26].

Furthermore, as illustrated in later sections, there are numerous questions related to some foundational issues in nonconvex optimization that have simple and natural formulations as SOS programs.

SOS programs as SDPs Sum of squares programs can be written as SDPs. The reason is the following theorem:

Theorem 1 *A polynomial* $p(x)$ *is SOS if and only if* $p(x) = z^T Q z$*, where* z *is a vector of monomials in the* x_i *variables,* $Q \in \mathcal{S}^N$ *and* $Q \succeq 0$*.*

In other words, every SOS polynomial can be written as a quadratic form in a set of monomials of cardinality N, with the corresponding matrix being positive semidefinite. The vector of monomials z (and therefore N) in general depends on the degree and sparsity pattern of $p(x)$. If $p(x)$ has n variables and total degree $2d$, then z can always be chosen as a subset of the set of monomials of degree less than or equal to d, of cardinality $N = \binom{n+d}{d}$.

Example 2 *Consider again the polynomial from Example 1. It has the representation*

$$p(x_1,x_2) = \frac{1}{6}\begin{bmatrix} 1 \\ x_2 \\ x_2^2 \\ x_1 \end{bmatrix}^T \begin{bmatrix} 6 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ x_2 \\ x_2^2 \\ x_1 \end{bmatrix}$$

and the matrix in the expression above is positive semidefinite.

In the representation $f(x) = z^T Q z$, for the right- and left-hand sides to be identical, all the coefficients of the corresponding polynomials should be equal. Since Q is simultaneously constrained by linear equations and a positive semidefiniteness condition, the problem can be easily seen to be directly equivalent to an SDP feasibility problem in the standard primal form (2).

Given a SOS program, we can use the theorem above to construct an equivalent SDP. The conversion step is fully algorithmic, and has been implemented, for instance, in the SOSTOOLS [36] software package. Therefore, we can in principle directly apply all the available numerical methods for SDP to solve SOS programs.

SOS and convexity The connection between sum of squares decompositions and convexity can be traced back to the work of N. Z. Shor [39]. In this 1987 paper, he essentially outlined the links between Hilbert's 17th problem and a class of convex bounds for unconstrained polynomial optimization problems. Unfortunately, the approach went mostly unnoticed for several years, probably due to the lack of the convenient framework of SDP.

3. Algebra and optimization

A central theme throughout convex optimization is the idea of *infeasibility certificates* (for instance, in LP via Farkas' lemma), or equivalently, *theorems of the alternative*. As we will see, the key link relating algebra and optimization in this approach is the fact that infeasibility can *always* be certified by a particular algebraic identity, whose solution is found via convex optimization.

We explain some of the concrete results in Theorem 5, after a brief introduction to two algebraic concepts, and a comparison with three well-known infeasibility certificates.

Ideals and cones For later reference, we define here two important algebraic objects: the *ideal* and the *cone* associated with a set of polynomials:

Definition 2 *Given a set of multivariate polynomials $\{f_1, \ldots, f_m\}$, let*

$$\mathbf{ideal}(f_1, \ldots, f_m) := \left\{ f \mid f = \sum_{i=1}^{m} t_i f_i, \quad t_i \in \mathbb{R}[x] \right\}.$$

Definition 3 *Given a set of multivariate polynomials $\{g_1, \ldots, g_m\}$, let*

$$\mathbf{cone}(g_1, \ldots, g_m) := \left\{ g \mid g = s_0 + \sum_{\{i\}} s_i g_i + \right.$$
$$\left. + \sum_{\{i,j\}} s_{ij} g_i g_j + \sum_{\{i,j,k\}} s_{ijk} g_i g_j g_k + \cdots \right\},$$

where each term in the sum is a squarefree product of the polynomials g_i, with a coefficient $s_\alpha \in \mathbb{R}[x]$ that is a sums of squares. The sum is finite, with a total of $2^m - 1$ terms, corresponding to the nonempty subsets of $\{g_1, \ldots, g_m\}$.

These algebraic objects will be used for deriving *valid inequalities*, which are logical consequences of the given constraints. Notice that by construction, every polynomial in $\mathbf{ideal}(f_i)$ vanishes in the solution set of $f_i(x) = 0$. Similarly, every element of $\mathbf{cone}(g_i)$ is clearly nonnegative on the feasible set of $g_i(x) \geq 0$.

The notions of *ideal* and *cone* as used above are standard in real algebraic geometry; see for instance [5]. In particular, the cones are also referred to as a *preorders*. Notice that as geometric objects, ideals are affine sets, and cones are closed under convex combinations and nonnegative scalings (*i.e.*, they are actually cones in the convex geometry sense). These convexity properties, coupled with the relationships between SDP and SOS, will be key for our developments in the next section.

Infeasibility certificates If a system of equations does not have solutions, how do we *prove* this fact? A very useful concept is that of *certificates*, which are formal algebraic identities that provide irrefutable evidence of the inexistence of solutions.

We briefly illustrate some well-known examples below. The first two deal with linear systems and polynomial equations over the complex numbers, respectively.

Theorem 2 (Range/kernel)

$$Ax = b \quad \textit{is infeasible}$$
$$\Updownarrow$$
$$\exists\, \mu \;\; s.t. \;\; A^T\mu = 0, \; b^T\mu = -1.$$

Theorem 3 (Hilbert's Nullstellensatz) *Let $f_i(z), \ldots, f_m(z)$ be polynomials in complex variables $z_1, \ldots, z_n$. Then,*

$$f_i(z) = 0 \quad (i = 1, \ldots, m) \quad \textit{is infeasible in } \mathbb{C}^n$$
$$\Updownarrow$$
$$-1 \in \mathbf{ideal}(f_1, \ldots, f_m).$$

Each of these theorems has an "easy" direction. For instance, for the first case, given the multipliers μ the infeasibility is obvious, since

$$Ax = b \quad \Rightarrow \quad \mu^T Ax = \mu^T b \quad \Rightarrow \quad 0 = -1,$$

which is clearly a contradiction.

The two theorems above deal only with the case of *equations*. The inclusion of inequalities in the problem formulation poses additional algebraic challenges, because we need to work on an *ordered* field. In other words, we need to take into account special properties of the *reals*, and not just the complex numbers.

For the case of *linear* inequalities, LP duality provides the following characterization:

Theorem 4 (Farkas lemma)

$$\left\{ \begin{array}{rcl} Ax + b & = & 0 \\ Cx + d & \geq & 0 \end{array} \right. \quad \textit{is infeasible}$$
$$\Updownarrow$$
$$\exists\, \lambda \geq 0,\ \mu \ \ s.t. \ \left\{ \begin{array}{rcl} A^T\mu + C^T\lambda & = & 0 \\ b^T\mu + d^T\lambda & = & -1. \end{array} \right.$$

Although not widely known in the optimization community until recently, it turns out that similar certificates do exist for *arbitrary* systems of polynomial equations and inequalities over the reals. The result essentially appears in this form in [5], and is due to Stengle [40].

Theorem 5 (Positivstellensatz)

$$\left\{ \begin{array}{rcll} f_i(x) & = & 0, & (i = 1, \ldots, m) \\ g_i(x) & \geq & 0, & (i = 1, \ldots, p) \end{array} \right.$$

is infeasible in $\mathbb{R}^n$ if and only if

$$\exists\, F(x), G(x) \in \mathbb{R}[x]$$

such that

$$\left\{ \begin{array}{l} F(x) + G(x) = -1 \\ F(x) \in \mathbf{ideal}(f_1, \ldots, f_m) \\ G(x) \in \mathbf{cone}(g_1, \ldots, g_p). \end{array} \right.$$

The theorem states that for every infeasible system of polynomial equations and inequalities, there exists a simple algebraic identity that directly certifies the inexistence of real solutions. By construction, the evaluation of the polynomial $F(x) + G(x)$ at any feasible point should produce a nonnegative number. However, since this expression is identically equal to the polynomial -1, we arrive at a contradiction. Remarkably, the Positivstellensatz holds under *no assumptions* whatsoever on the polynomials.

The use of the German word "Positivstellensatz" is standard in the field, and parallels the classical "Nullstellensatz" (roughly, "theorem of the zeros") obtained by Hilbert in 1901 and mentioned above.

In the worst case, the degree of the infeasibility certificates $F(x), G(x)$ could be high (of course, this is to be expected, due to the NP-hardness of the original question). In fact, there are a few explicit counterexamples where large degree refutations are necessary [16]. Nevertheless, for many problems of practical interest, it is often the case that it is possible to prove infeasibility using relatively low-degree certificates. There is significant numerical evidence that this is the case, as indicated by the large number of practical applications where SDP relaxations based on these techniques have provided solutions of very high quality.

Of course, we are concerned with the effective computation of these certificates. For the cases of Theorems 2–4, the corresponding refutations can be obtained using either linear algebra, linear programming, or Groebner bases techniques (see [8] for a superb introduction to Groebner bases).

For the Positivstellensatz, we notice that the cones and ideals as defined above are always *convex sets* in the space of polynomials. A key consequence is that the conditions in Theorem 5 for a certificate to exist are therefore *convex*, regardless of any convexity property of the original problem. Even more, the same property holds if we consider only bounded-degree sections, *i.e.*, the intersection with the set of polynomials of degree less than or equal to a given

Degree \ Field	Complex	Real
Linear	*Range/Kernel* Linear Algebra	*Farkas Lemma* Linear Programming
Polynomial	*Nullstellensatz* Bounded degree: Linear Algebra Groebner bases	*Positivstellensatz* Bounded degree: SDP

Table 1: Infeasibility certificates and associated computational techniques.

number D. In this case, the conditions in the P-satz have *exactly* the form of a SOS program! Of course, as discussed earlier, this implies that we can find bounded-degree certificates, by solving semidefinite programs. In Table 1 we present a summary of the infeasibility certificates discussed, and the associated computational techniques.

Example 3 *Consider again the system (1). We will show that it has no solutions* $(x_1, x_2) \in \mathbb{R}^2$. *By the P-satz, the system is infeasible if and only if there exist polynomials* $t_1, s_0, s_1, s_2, s_{12} \in \mathbb{R}[x_1, x_2]$ *that satisfy*

$$\underbrace{f_1 \cdot t_1}_{\mathbf{ideal}(f_1)} + \underbrace{s_0 + s_1 \cdot g_1 + s_2 \cdot g_2 + s_{12} \cdot g_1 \cdot g_2}_{\mathbf{cone}(g_1,g_2)} \equiv -1, \tag{3}$$

where s_0, s_1, s_2, *and* s_{12} *are SOS.*

A SOS relaxation is obtained by looking for solutions where all the terms in the left-hand side have degree less than or equal to D. *For each fixed integer* $D > 0$ *this can be tested by semidefinite programming. For instance, for* $D = 4$ *we find the certificate*

$$t_1 = -3x_1^2 + x_1 - 3x_2^2 + 6x_2 - 2,$$

$$s_1 = 3, \qquad s_2 = 1, \qquad s_{12} = 0,$$

$$\begin{aligned} s_0 &= 3x_1^4 + 2x_1^3 + 6x_1^2x_2^2 - 6x_1^2x_2 - x_1^2 - x_1x_2^2 + \\ &\qquad +3x_2^4 + 2x_2^3 - x_2^2 - 3x_2 + 3 \\ &= \frac{1}{2} z^T \begin{bmatrix} 6 & -3 & -3 & 0 & 0 & -3 \\ -3 & 4 & 2 & 0 & 1 & 1 \\ -3 & 2 & 6 & -2 & 0 & -3 \\ 0 & 0 & -2 & 4 & -7 & 2 \\ 0 & 1 & 0 & -7 & 18 & 0 \\ -3 & 1 & -3 & 2 & 0 & 6 \end{bmatrix} z, \end{aligned}$$

where

$$z = \begin{bmatrix} 1 & x_2 & x_2^2 & x_1 & x_1x_2 & x_1^2 \end{bmatrix}^T.$$

The resulting identity (3) thus certifies the inconsistency of the system $\{f_1 = 0,\ g_1 \geq 0,\ g_2 \geq 0\}$.

As outlined in the preceding paragraphs, there is a direct connection going from general polynomial optimization problems to SDP, via P-satz infeasibility certificates. Pictorially, we have the following:

$$\begin{array}{c} \text{Polynomial systems} \\ \Downarrow \\ \text{P-satz certificates} \\ \Downarrow \\ \text{SOS programs} \\ \Downarrow \\ \text{SDP} \end{array}$$

Even though we have discussed only feasibility problems, there are obvious straightforward connections with optimization. By considering the emptiness of the sublevel sets of the objective function, sequences of converging bounds indexed by certificate degree can be directly constructed.

4. Further developments and applications

We have covered only the core elements of the SOS/SDP approach. Much more is known, and even more still remains to be discovered, both in the theoretical and computational ends. Some specific issues are discussed below.

Exploiting structure and numerical computation To what extent can the inherent structure in SOS programs be exploited for efficient computations? Given the algebraic origins of the formulation, it is perhaps not surprising to find that several intrinsic properties of the input polynomials can be profitably used, see [29]. In this direction, symmetry reduction techniques have been employed by Gatermann and Parrilo in [14] to provide novel representations for symmetric polynomials. Kojima, Kim and Waki [20] have recently presented some results for sparse polynomials. Parrilo [31] and Laurent [23] have analyzed the further simplifications that occur when the inequality constraints define a zero-dimensional ideal.

Other relaxations Lasserre [21, 22] has independently introduced a scheme for polynomial optimization dual to the one described here, but relying on Putinar's representation theorem for positive polynomials rather than on the P-satz. There are very interesting relationship between SOS-based methods and earlier relaxation and approximation schemes, such as Lovász-Schrijver and Sherali-Adams. Laurent [24] analyzes this in the specific case of 0-1 programming.

Implementations The software SOSTOOLS [36] is a free, third-party MATLAB[1] toolbox for formulating and solving general sum of squares programs. The related software GloptiPoly [17] is oriented toward global optimization problems. In their current version, both use the SDP solver SeDuMi [10] for numerical computations.

Approximation properties There are several important open questions regarding the provable quality of the approximations. In this direction, De Klerk and Pasechnik [11] have established some approximations guarantees of a SOS-based scheme for the approximation of the stability number of a graph. Recently, De Klerk, Laurent, and Parrilo [10] have shown that a related procedure based on a result by Pólya provides a polynomial-time approximation scheme (PTAS) for polynomial optimization over simplices.

Applications There are many exciting applications of the ideas described here. The descriptions that follow are necessarily brief; our main objective here is to provide the reader with some good starting points to this growing literature.

In systems and control theory, the techniques have provided some of the best available analysis and design methods, in areas such as nonlinear stability and robustness analysis [30, 28, 35], state feedback control [19], fixed-order controllers [18], nonlinear synthesis [37], and model validation [34]. Also, there have been interesting recent applications in geometric theorem proving [33] and quantum information theory [12, 13].

Acknowledgments: The author would like to thank Etienne de Klerk and Luís N. Vicente for their helpful comments and suggestions.

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Optimization with Polynomials and Fixed-Order Robust Controllers: A Design Example

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Abstract: With the help of an inverted pendulum example, we show how convex optimization over linear matrix inequalities can be used to design a robust H_∞ controller of fixed order, based on a purely algebraic approach and recent results on positive polynomials.

1. Introduction

The objective of this short note is to explain with the help of a simple example how *algebraic control techniques* can be combined with *convex optimization* to solve a potentially difficult control problem. The focus here is mainly on the numerical example, and the underlying theory is described elsewhere. We apply the methodology proposed in [4] and [5] to design a *fixed-order robust controller* minimizing H_∞ performance criteria for an *inverted pendulum system.*

Numerical experiments were carried out with the help of the Polynomial Toolbox 2.5 [8], the LMI interface YALMIP 2.1 [6], and the semidefinite programming solver SeDuMi 1.05 [10], under a Matlab 6.5 environment running on a Sun Solaris Sparc Blade 150 workstation.

2. Design with one measurement

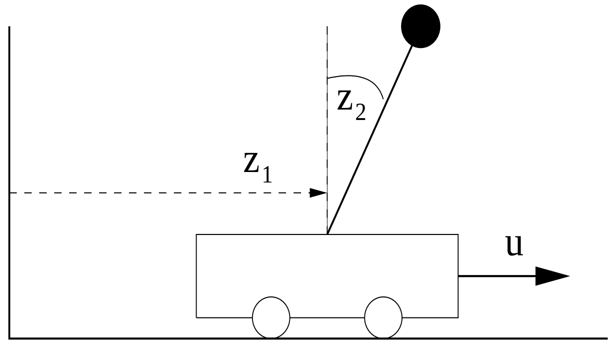

Figure 1: Inverted pendulum system.

We consider the inverted pendulum setup of figure 1, described in [7, Section 7], whose linearized transfer function from input u (dragging force) to output z_1 (card position) is given by the rational function $G(s) = G_1(s) = b_1(s)/a(s) = 2(s^2 - 10)/(s^2(s^2 - 20))$ in the complex Laplace variable s.

First we want to find a single-input-single-output (SISO) robust linear control law $u(s) = K_1(s)e(s)$ with $K_1(s) = y_1(s)/x(s)$ such that pendulum position $z_1(s)$ tracks a reference position $z_*(s)$, according to the classical error feedback scheme $e(s) = z_*(s) - z_1(s)$. In the H_∞ setting, closed-loop robustness is ensured if the sensitivity function $S_{11}(s) = 1/(1 + G_1(s)K_1(s))$ and the complementary sensitivity function $T_{11}(s) = 1 - S_{11}(s)$ have sufficiently small H_∞ norms [3]. Recall that

$$\|S\|_\infty = \sup_{s=j\omega,\,\omega\in\mathbb{R}} |S(s)|$$

denotes the peak value of the magnitude of rational transfer function S evaluated along the imaginary axis. The notation j, traditionally used in control systems engineering, refers to the imaginary unit $j = \sqrt{-1}$. Physically, a high value of $\|S\|_\infty$ attained at some $s_* = j\omega_*$ corresponds to a resonance peak at pulsation ω_*. Hence, roughly speaking, minimizing the H_∞ norm amounts to *removing resonance peaks* as much as possible.

We formulate our SISO H_∞ fixed-order controller design problem as follows: given plant polynomials[1] $a(s)$ and $b_1(s)$ of degree n (the plant order), find controller

[1] All the polynomials in this note are real coefficient polynomials of the complex Laplace variable s.

polynomials $x(s)$ and $y_1(s)$ of given degree m (the controller order) such that

$$\|S_{11}(s)\|_\infty = \left\| \frac{a(s)x(s)}{a(s)x(s)+b_1(s)y_1(s)} \right\|_\infty \leq M_{S_{11}}$$

where $M_{S_{11}}$ is a given upper bound. Here we do not enforce an upper bound on the H_∞ norm of $T_{11}(s)$. Practice reveals that in this case, it will be minimized by side-effect since $S_{11}(s) + T_{11}(s) = 1$.

3. Fixed-order controller design

In general, H_∞ controller design is *difficult* as soon as the *order of the controller to be found is strictly less than the order of the plant to be controlled.* We are not aware of any tractable numerical method to compute systematically such low-order H_∞ controllers. In several applications such as embedded control systems for the aerospace industry, low controller order is a fundamental requirement because it means easy and light implementation of the control law.

In [4] and [5], results on *positive polynomials* were used to come up with a *linear matrix inequality* (LMI) formulation of fixed-order robust and H_∞ controller design, in the algebraic, or polynomial framework initiated in [2]. However, the LMI formulation is potentially conservative, meaning that we have no guarantee to solve the fixed-order controller design problem even if we know that a solution exists. As explained in [4], the key ingredient in the design procedure resides in the choice of a *central polynomial*, or desired nominal closed-loop characteristic polynomial. The name central polynomial is used to emphasize the fact that this is a reference characteristic polynomial around which design is carried out, see [4]. The main result of [5] can be summarized as follows.

Suppose that we are given a set of polynomials $n_i^k(s)$, $d_i^k(s)$ as well as a set of positive real numbers γ^k for $i, k = 1, 2, \ldots$ We seek polynomials $x_i(s)$ of given degrees such that

$$\left\| \frac{\sum_i n_i^k(s)x_i(s)}{\sum_i d_i^k(s)x_i(s)} \right\|_\infty < \gamma^k, \quad k = 1, 2, \ldots \qquad (1)$$

Following [5], we define $c(s)$ as a central polynomial of the same degree δ as the polynomials

$$n^k(s) = \gamma \sum_i d_i^k(s)x_i(s) + \sum_i n_i^k(s)x_i(s)$$

and

$$d^k(s) = \gamma \sum_i d_i^k(s)x_i(s) - \sum_i n_i^k(s)x_i(s).$$

We also define the 2-by-2 polynomial matrices $N^k(s) = \operatorname{diag}\{n^k(s),\, n^k(s)\}$, $D^k(s) = \operatorname{diag}\{d^k(s),\, d^k(s)\}$, $C(s) = [c(s)\ c(s);\ -c(s)\ c(s)]$ whose coefficient matrices, corresponding to increasing powers of the variable s, are gathered in the block matrices N^k, D^k, and C respectively. Note that the matrices N^k and D^k are linear in the coefficients of the polynomials $x_i(s)$. With these notations, results on positive polynomial matrices are invoked in [5] to derive the following solution to the H_∞ fixed-order controller design problem.

Theorem 1 *Given a central polynomial $c(s)$, if the matrix inequalities*

$$\begin{aligned} &(N^k)^T C + C^T N^k - H(P_n^k) \succ 0, \\ &(D^k)^T C + C^T D^k - H(P_d^k) \succ 0, \quad k = 1, 2, \ldots \end{aligned} \qquad (2)$$

are feasible, then H_∞ specifications (1) are met. This is a convex LMI problem in the coefficients of polynomials $x_i(s)$ and symmetric matrices P_n^k and P_d^k.

The central polynomial $c(s)$ plays the role of a target closed-loop characteristic polynomial around which the design is carried out. Sensible strategies for the choice of the central polynomial $c(s)$ are discussed in [4] and [5].

4. Application to the inverted pendulum problem

Suppose we are seeking a third-order ($m = 3$) controller for our fourth-order ($n = 4$) inverted pendulum plant described in Section 2.. Using results of Section 3., in order to derive an LMI formulation for this design problem, we must choose a suitable central polynomial. Following rules of thumb explained in [4] and [5], we found by an error-and-trial iterative procedure the following seventh-degree ($n + m = 7$) central polynomial $c(s) = (s + 20)(s + 10)(s + 3)^2(s + 1)^2(s + 0.01)$.

With the upper bound $M_{S_{11}} = 20$ we solve LMI problem (2) to obtain the third degree controller polynomials $x(s) = -2441.0 - 929.36s + 17.699s^2 + s^3$ and $y_1(s) = -0.74001 - 32.610s + 1110.4s^2 + 349.23s^3$ yielding $\|S_{11}(s)\|_\infty = 14.610$, $\|T_{11}(s)\|_\infty = 14.459$.

Typical acceptable values for these H_∞ norms are between 1 and 2 (see [3]), so we are *very far from a sensible robust design.* It can be checked that the system response features unacceptable undershoot and overshoot which would probably exceed physical capabilities of the plant. This could be expected. Indeed, as pointed out in [7], the open-loop system is both unstable and non-minimum phase, and the unstable zero occurs at a lower frequency than the unstable pole. It means that any stabilizing controller that can be designed for this plant will have very poor performance and robustness properties. As pointed out in [7], it is easy to gain some appreciation of the difficulty of the control problem when only the pendulum position z_1 is measured, by *trying to balance a broom with both eyes shut.*

5. Design with two measurements

In order to improve closed-loop performance and robustness, we consider in this section that the *angle* z_2 of the pendulum is also measured and available for feedback, along with the pendulum *position* z_1. Therefore we will have to solve a single-input-multi-output (SIMO) design problem.

Let $G(s) = [G_1(s); G_2(s)]$ denote the SIMO open-loop transfer function, where $G_1(s)$, as previously, is the transfer function from $u(s)$ to $z_1(s)$, and $G_2(s) = b_2(s)/a(s) = -2s^2/(s^2(s^2-20))$ is the transfer function from $u(s)$ to $z_2(s)$.

As in [7] we consider the feedback configuration $u(s) = K_1(s)e(s) - K_2(s)z_2(s)$, with $e(s) = z_*(s) - z(s)$. The feedback matrix $K(s) = [K_1(s) \quad K_2(s)]$ consists of $K_1(s) = y_1(s)/x(s)$ as previously and $K_2(s) = y_2(s)/x(s)$. The 2-by-2 sensitivity functions are now given by $S(s) = (I + G(s)K(s))^{-1}$ and $T(s) = I - S(s)$. Let $S_{ij}(s)$ and $T_{ij}(s)$ denote entries (i,j) in $S(s)$ and $T(s)$, respectively.

We formulate our SIMO H_∞ design problem as follows: given plant polynomials $a(s)$, $b_1(s)$, and $b_2(s)$ of degree n, find controller polynomials $x(s)$, $y_1(s)$, and $y_2(s)$ of given degree m such that

$$\begin{array}{l} \|T_{11}(s)\|_\infty = \|\frac{b_1(s)y_1(s)}{a(s)x(s)+b_1(s)y_1(s)+b_2(s)y_2(s)}\|_\infty \leq M_{T_{11}}, \\ \|T_{22}(s)\|_\infty = \|\frac{b_2(s)y_2(s)}{a(s)x(s)+b_1(s)y_1(s)+b_2(s)y_2(s)}\|_\infty \leq M_{T_{22}}, \end{array}$$

where $M_{T_{11}}$ and $M_{T_{22}}$ are given upper bounds.

Suppose we are seeking a second-order ($m = 2$) controller. After various trials consisting in decreasing the upper bounds while moving apart the roots of the central polynomial, we come up with the central polynomial $c(s) = (s+20)^2(s+1)^2(s+0.1)^2$ and the upper bounds $M_{T_{11}} = 2$ and $M_{T_{22}} = 3$. We refer the interested reader to [4] for a description of this heuristic iterative procedure. Upon solving LMI problem (2) we obtain the second degree controller polynomials $x(s) = 116.35 + 145.90s + s^2$, $y_1(s) = -1.6205 - 33.693s - 38.843s^2$, and $y_2(s) = -1918.5 - 2764.9s - 496.12s^2$, yielding $\|S_{11}(s)\|_\infty = 1.1050$ and $\|T_{11}(s)\|_\infty = 1.1419$.

Comparing to the results of the previous section, the improvement brought by the second measurement is significant, since the H_∞ norms of sensitivity and complementary sensitivity functions are now significantly less than 2 with our second-order controller. A comparable performance was achieved in [7] with a more complicated fourth-order controller.

6. Conclusion

In this note we have applied the polynomial/LMI approach described in [4] and [5] to design a fixed-order robust controller satisfying H_∞ performance specifications for an inverted pendulum system.

A promising research direction is the study of *numerical properties* (computational complexity, numerical stability) of algorithms tailored to solve *LMI problems coming from polynomial positivity conditions*. As shown in [1], the Hankel or Toeplitz structure can be exploited to design fast algorithms to solve Newton steps in barrier schemes and interior-point algorithms. Numerical stability is also a concern, since it is well-known for example that Hankel matrices are exponentially ill-conditioned. Alternative polynomial bases such as Chebyshev or Bernstein polynomials may prove useful.

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Bulletin

1. Workshop Announcements

Eighth SIAM Conference on Optimization
Sunday to Wednesday, May 15-18, 2005,
Stockholm, Sweden
`http://www.siam.org/meetings/op05`

The Eighth SIAM Conference on Optimization, sponsored by the SIAM Activity Group on Optimization, will feature the latest research in theory, algorithms, and applications in optimization problems. In particular, it will emphasize large-scale problems and will feature important applications in networks, manufacturing, medicine, biology, finance, aeronautics, control, operations research, and other areas of science and engineering. The conference brings together mathematicians, operations researchers, computer scientists, engineers, and software developers; thus it provides an excellent opportunity for sharing ideas and problems among specialists and users of optimization in academia, government, and industry.

The themes of the conference include, but are not limited to: large-scale nonlinear programming; large-scale linear programming; simulation-based optimization; optimization in medicine and biology; stochastic programming; optimization in finance; semidefinite programming; computational optimization frameworks.

The organizing committee is formed by 12 members and co-chaired by A. Forsgren (Royal Institute of Technology, Sweden) and H. Wolkowicz (University of Waterloo).

The invited plenary speakers are: D. Bienstock (Columbia University), M. C. Ferris (University of Wisconsin), O. Ghattas (Carnegie Mellon University), J. Gondzio (University of Edinburgh), M. Kojima (Tokyo Institute of Technology), M. Laurent (CWI, The Netherlands), A. Shapiro (Georgia Institute of Technology), and A. Sofer (George Mason University).

A number of minisymposia have already been submitted, in topics like:

- Advances in Large Sparse Nonlinear Programming.
- Applications in Finance.
- Applications in Medicine and Biology.
- Computational Biology.
- Applications in VLSI Design.
- Complementarity Problems.
- Derivative-Free Optimization.
- Optimization in Aerospace Engineering.
- Implementation and Robustness for Interior-Point Methods.
- Interior-Point Methods for Very Large Problems.
- Multidisciplinary Optimization.
- Network Optimization.
- Nonlinear Semidefinite Programming
- Progress in Nonlinear Optimization and Filter Methods.
- Simulation Based Design Optimization.

Workshop on Optimization in Finance
July 5-9, 2005, International Center for Mathematics,
University of Coimbra, Portugal
`http://www.mat.uc.pt/tt2005/of`

Optimization models and methods play an increasingly important role in financial decision making. Many problems in quantitative finance, originated from asset allocation, risk management, derivative pricing, and model fitting, are now routinely and efficiently solved using modern optimization techniques. This workshop will bring together researchers in the rapidly growing field of financial optimization and intends to provide a forum for innovative models and methods on new topics, novel approaches to well-known problems, success stories, and computational studies in this exciting field. Participants are encouraged to present and discuss their recent work and new, possibly controversial, approaches are particularly welcome.

The targeted audience for this workshop includes graduate students and faculty members working in applied mathematics, operations research, and economics, who have been interested in mathematical finance or plan to do so. The workshop will also be attractive for those doing quantitative modeling in the financial market.

A one-day short-course, intended for optimization researchers interested in quantitative finance as well as finance researchers and practitioners interested in optimization models and methods, will precede the scientific program of the workshop. Invited and contributed presentations will be scheduled during the remaining three days.

The invited plenary speakers are: J. R. Birge (University of Chicago), T. F. Coleman (Cornell University), H. Konno (Chuo University, Japan), J. M. Mulvey (Princeton University), R. T. Rockafellar (University of Washington), N. Touzi (CREST, France), and S. A. Zenios (University of Cyprus). The short course is given by R. H. Tütüncü (Carnegie Mellon University) and S. Uryasev (University of Florida).

Summer School on Geometric and Algebraic Approaches for Integer Programming
July 11–15, 2005, International Center for Mathematics, University of Lisbon, Portugal
`http://www.mat.uc.pt/tt2005/ss`

The School is composed by five set of lectures, designed to introduce young researchers to the more recent advances on geometric and algebraic approaches for integer programming. Each set of lectures will be about six hours long. They will provide the background, introduce the theme, describe the state-of-the-art, and suggest practical exercises. The organizers will try to provide a relaxed atmosphere with enough time for discussion.

Integer programming is a field of optimization with recognized scientific and economical relevance. The usual approach to solve integer programming problems is to use linear programming within a branch-and-bound or branch-and-cut framework, using whenever possible polyhedral results about the set of feasible solutions. Alternative algebraic and geometric approaches have recently emerged that show great promise. In particular, polynomial algorithms for solving integer programs in fixed dimension have recently been developed. This is a hot topic of international research, and the School will be an opportunity to bring up-to-date knowledge to young researchers.

The School is composed by five lectures, designed to introduce young researchers to the more recent advances on geometric and algebraic approaches for integer programming. Each lecture will be about three hours long. They will provide the background, introduce the theme, describe the state-of-the-art, and suggest practical exercises. The organizers will try to provide a relaxed atmosphere with enough time for discussion.

The invited lecturers are: A. Barvinok (University of Michigan), G. Cornuéjols (Carnegie Mellon University), F. Eisenbrand (Max-Planck-Institut), J. de Loera (University of California, Davis), and R. Weismantel (Otto-von-Guericke Univ. Magdeburg).

22nd IFIP TC 7 Conference on System Modeling and Optimization
July 18-22, 2005, Turin, Italy
`http://www2.polito.it/eventi/ifip2005`

The themes of the conference are mathematical models methods and algorithms in optimization, identification, simulation and their applications:

- Optimization; optimization with PDE constraints; structural systems optimization; algorithms for linear and nonlinear programming; stochastic optimization; control and game theory; combinatorial and discrete optimization.
- Identification and inverse problems; fault detection; shape identification.
- Complex systems; stability and sensitivity analysis; neural networks; fractal and chaos; reliability.
- Computational techniques in distributed systems and in information processing environments; transmission of information in complex systems; data base design.
- Applications of optimization techniques and of computational methods to scientific and technological areas (such as medicine, biology, economics, finances, aerospace and aeronautics, *etc.*).

The invited plenary speakers are: R. Fletcher (University of Dundee), A. Forsgren (Royal Institute of Technology, Sweden), D. M. Frangopol (University of Colorado), W. Hager (University of Florida), J. Mayer (University of Zurich), J. Nocedal (Northwestern University), A. Quarteroni (EPFL, Switzerland), H. M. Soner (Koc University, Turkey), G. Uhlmann (University of Washington), and R. Zecchina (ICTP, Italy).

Workshop on Optimization in Medicine
July 20-22, 2005, International Center for Mathematics, University of Coimbra, Portugal
`http://www.mat.uc.pt/tt2005/om`

The study of computing in medical applications has opened many challenging issues and problems for both the medical computing and mathematical communities. This workshop is intended to foster communication and collaboration between researchers in the medical computing community and researchers working in applied mathematics and optimization.

Mathematical techniques (continuous and discrete) are playing a key role with increasingly importance in understanding several fundamental problems in medicine.

For instance, mathematical theory of nonlinear dynamics and discrete optimization has been used to predict epileptic seizures. Next to stroke, epilepsy is among the most common disorders of the nervous system. Measures derived from the theory of nonlinear dynamics and discrete optimization techniques are used for prediction of impending epileptic seizures from analysis of multielectrode electroencephalographic (EEG) data.

Several examples of the use of mathematics in medicine can be found in recent cancer research. Sophisticated mathematical models and algorithms have been used for generating treatment plans for radionuclide implant and external beam radiation therapy. With Gamma Knife treatment, for example, optimization techniques have been used to automate the treatment planning process.

Optimization has been used to address a variety of medical image registration problems. In particular, specialized mathematical programming techniques have been used in a variety of domains including the rigid alignment of primate autoradiographs and the non-rigid registration of cortical anatomical structures as seen in MRI.

The invited presentations will be complemented by sessions of contributed talks. The invited plenary speakers are: M. C. Ferris (University of Wisconsin), H. W. Hamacher (University of Kaiserslautern), L. D. Iasemidis (Arizona State University), J. P. Kaipio (University of Kuopio), E. K. Lee (Georgia Institute of Technology), and A. Rangarajan (University of Florida).

Workshop on PDE Constrained Optimization
July 26-29, 2005, International Center for Mathematics, Tomar, Portugal
`http://www.mat.uc.pt/tt2005/pde`

Optimization problems governed by partial differential equation (PDE) constraints arise in many important applications. Progress in computational and applied mathematics combined with the availability of rapidly increasing computer power steadily enlarges the range of applications that can be simulated numerically and for which optimization tasks, such as optimal design, parameter identification, and control are being considered. For most of these optimization problems, simple approaches combining off-the-shelf PDE solvers and optimization algorithms often lack robustness or can be very inefficient.

Successful solution approaches have to overcome challenges arising from, *e.g.*, the increasing complexity of applications and their mathematical models, the influence of the underlying infinite dimensional problem structure on optimization algorithms, and the interaction of PDE discretization and optimization.

This workshop will combine a wide range of topics important to PDE constrained optimization in an integrated approach, fusing techniques from a number of mathematical disciplines including functional analysis, optimal control theory, numerical optimization, numerical PDEs, and numerical analysis and application specific structures.

A short course will be offered on the first day of the workshop by F. Tröltzsch (Technical University of Berlin) and M. Heinkenschloss (Rice University).

Invited and contributed presentations will be scheduled during the remaining three days. The invited plenary speakers are: M. D. Gunzburger (Florida State University), R. H. W. Hoppe (University of Augsburg), K. Kunisch (University of Graz), G. Leugering (Univ. Erlangen-Nrnb.), A. T. Patera (MIT), R. Rannacher (University of Heidelberg), and E. W. Sachs (University of Trier).

2. Other Announcements

SIAM Activity Group on Optimization Prize
CALL FOR NOMINATIONS

The SIAM Activity Group on Optimization Prize (SIAG/OPT Prize) will be awarded at the SIAM Conference on Optimization to be held May 15-18, 2005, in Stockholm, Sweden.

The SIAG/OPT Prize, established in 1992, is awarded to the author(s) of the most outstanding paper, as determined by the prize committee, on a topic in optimization published in English in a peer-reviewed journal. The award period is the four calendar years preceding the year of the conference.

Eligibility

Candidate papers must be published in English in a peer-reviewed journal bearing a publication date within the award period. Thus, to be eligible for the prize, a paper must appear with a publication date in the 2001-2004 calendar years. Candidate papers must contain significant research contributions to the field of optimization, as commonly defined in the mathematical literature, with direct or potential applications.

Description of the Award

The award will consist of a plaque and a certificate containing the citation. At least one of the prize recipients is expected to attend the award ceremony and to present the paper at the conference.

Nominations

A letter of nomination, including a citation of the paper, should be sent by January 15, 2005, to:

SIAM Activity Group on Optimization Prize
Professor Robert Vanderbei, Chair
c/o J. M. Littleton
SIAM
3600 University City Science Center
Philadelphia, PA 19104-2688
USA

E-mail: **littleton@siam.org**
Telephone: 215-382-9800
Fax: 215-386-7999

Selection Committee

The members of the selection committee are: Robert Vanderbei (chair), Princeton University; Aharon Ben-Tal, Technion; Adrian Lewis, Simon Fraser University; S. Thomas McCormick, University of British Columbia; and Yinyu Ye, Stanford University.

COAP 2003 Best Paper Award

The journal, Computational Optimization and Applications (COAP), has announced that Jeff Linderoth, Lehigh University, and Stephen Wright, University of Wisconsin-Madison, have won the COAP 2003 Best Paper Award for *Decomposition Algorithms for Stochastic Programming on a Computational Grid*, published in COAP, Volume 24, pp. 207-250. This paper demonstrates the vast potential of harnessing the computational capabilities of the millions of processors connected through the internet. Using over one thousand computers spread across the United States and in Europe, the authors describe how they solved many challenging problems. In one case, they solve a flight mobilization model involving billions of decisions in just over a day when conventional computing would have required over a year. To achieve these results, they provide new solution methods that work asynchronously and fit the heterogeneous, dynamic, and unreliable computing environment of widely dispersed machines that exists today. Their work provides a solid platform for further developments in exploiting the power of the computational grid.

Special Issue of ETNA on Saddle Point Problems: Numerical Solution and Applications

The journal Electronic Transactions on Numerical Analysis (ETNA) will devote a special issue to the solution of saddle point problems. These problems arise in systems of PDEs with conservation laws (including Stokes and Navier-Stokes equations, incompressible linear elasticity, magnetostatics, *etc.*), in constrained optimization problems, in generalized least squares problems, and elsewhere. Such problems pervade computational science and engineering, and their efficient numerical solution is of paramount importance. However, to compute accurate solutions to saddle point problems at a reasonable cost has proved difficult. As a consequence, a significant effort has been devoted to define proper formulations, (stabilized) discretizations, and fast solution methods for discretized saddle point problems and their generalizations.

The present issue aims to attract papers on the continuous and discrete formulation of saddle point problems in all aspects of computational science, both for specific problems and in general, and on efficient solution techniques for the resulting systems of equations. Papers describing novel applications leading to saddle point problems are also of interest.

Manuscripts should be submitted electronically to one of the special issue editors listed below. The deadline for submission of papers is December 15, 2004. All submissions will undergo the standard refereeing process used for regular ETNA papers. The issue is expected to appear by the end of 2005. For more information on ETNA, see the web site `http://etna.mcs.kent.edu`.

If you have any questions, please contact the editors of the special issue:

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Chairman's Column

This is my first column for Views-and-News as the Chair of SIAG/OPT. I am honored to have been elected, and along with the other newly-elected board members (Vice Chair Bob Vanderbei, Program Director Sven Leyffer and Secretary/Treasurer Kees Roos) look forward to serving the organization. I am delighted to report that Luís Vicente has agreed to continue as the editor of SIAG/Optimization Views-and-News, having taken over that position following the tragic death of Jos Sturm last year.

I would first like to thank our former Chair Henry Wolkowicz and the other outgoing board members (Vice Chair Philippe Toint, Program Director Anders Forsgren and Secretary/Treasurer Natalia Alexandrov) for the outstanding leadership they have provided over the past three years. SIAG/OPT is particularly indebted to Natalia for her work maintaining the Activity Group's web site. This responsibility has recently been taken over by Sven Leyffer, and the SIAG/OPT home page is now at the URL `http://www.mcs.anl.gov/~leyffer/siagopt`.

Henry and Anders have left their board responsibilities behind, but are now very busy as Co-Chairs of the Organizing Committee for the 8th SIAM Conference on Optimization to be held in Stockholm, Sweden on May 15-18, 2005. The triennial SIOPT conference is now well established as one of the foremost international meetings on optimization. This year's meeting locale is an appropriate indicator of the very international make-up of our organization — over 1/3 of the members of SIAG/OPT are outside of the USA. The meeting combines an outstanding technical program with a beautiful location and should be well attended by SIAG/OPT members and nonmembers alike.

We all know that optimization has many important applications in business, engineering, logistics, medicine, economics and other areas. In recent years there have also been some very significant, but perhaps less well-known, applications in the solution of long-open problems in mathematics. One such application that I would like to briefly describe is the solution of the well-known Kepler conjecture by Thomas Hales and his student Sam Ferguson, first announced in 1998. Kepler's conjecture, which dates to the early 17th century, asserts that the densest packing of equal-sized spheres in three-dimensional space is achieved by the "hexagonal close packing" (HCP), corresponding to how oranges are stacked in a grocery store (or, of some historical importance, cannonballs are stacked on the deck of a ship). A nicely written account of the history of the Kepler conjecture, including a description of Hales' proof, is in the book *Kepler's Conjecture* by George Szpiro, Wiley, 2003. Chapter 13 of Szpiro's book is entitled "Simplex, Cplex, and Symbolic Mathematics" The first sentence in the chapter reads "The proof of Kepler's conjecture is basically an optimization problem," and a few pages later the following paragraph appears:

> "A typical example of Hales and Ferguson's problem had between one hundred and two hundred variables, and between one thousand and two thousand constraints. The variables in the linear programs were angles, volumes, and distances. The constraints expressed the conditions on lengths and angles so that only those packings that could actually exist were considered. Nearly one hundred thousand such problems had to be solved in the proof. In 98 percent of the five thousand nets that Hales investigated, that method worked."

Szpiro goes on to explain that in more difficult cases Hales and Ferguson had to resort to mixed-integer problems so as to obtain tighter bounds on the maximum "score" attainable by a given geometric configuration. By eventually showing that no configuration corresponding to a packing could attain a score higher than that attained by the HCP, Hales and Ferguson proved Kepler's conjecture. Hales' approach to the problem includes many novel ideas and complex details, but it is obvious that without the availability of fast, reliable software for linear and mixed-integer programming (in this case CPLEX) the proof attempt would have failed. For a more detailed description of Hales' approach see the article *Bounds for the local density of sphere packings and the Kepler conjecture* by Jeff Lagarias, Discrete and Computational Geometry, 27 (2002), pp. 165–193.

The Hales and Ferguson proof is still considered controversial due to the enormous amount of computation involved. The situation is not unlike that of the first proof of the four-color theorem, by Appel and Haken in 1976. The Appel-Haken proof involved so much computation that it is believed to have never been independently verified. However, a conceptually similar but greatly simplified proof by Robertson, Sanders, Seymour and Thomas, announced in 1995, reduces the computation required to the point that the proof can be relatively easily checked. It is entirely possible that a simpler proof of Kepler's conjecture will be found in years to come, but in all likelihood the proof will continue to build on optimization as a cornerstone.

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Comments from the Editor

This issue of the SIAG/Optimization Views-and-News contains three expository articles about polynomial optimization problems, in particular sum of squares programs, and their relaxations via semidefinite programming. I was very pleased to work with the guest editor, Etienne de Klerk, who had been invited by Jos Sturm to edit such a issue. I would like to thank the authors for their (non trivial) efforts in presenting technical material in a form accessible to the whole SIAG/OPT community.

I take this opportunity, once again, to ask for contributions like expository articles on interesting topics (ranging from applications or case studies to theory or software), announcements of awards and events, and book and software releases.

I cannot possibly miss this chance to wish a successful term to our new board members, Kurt Anstreicher, Bob Vanderbei, Sven Leyffer, and Kees Roos.

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